Lecture 8: Solutions to the Strong CP Problem and the Axion

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Lecture date: April 10, 2023

Last update: June 6, 2023

In the last lecture we saw that the so-called θ-term

$$\mathcal{L}_\theta = \frac{\overline{\theta} g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} = \frac{\overline{\theta} g_s^2}{16\pi^2} \operatorname{Tr} G_{\mu\nu} \widetilde{G}^{\mu\nu} =: \frac{\overline{\theta} g_s^2}{32\pi^2} G\widetilde{G}. \tag{1}$$

has to be included in the QCD Lagrangian. The parameter $\overline{\theta}$ has two contributions:

$$\overline{\theta} = \theta_{\text{QCD}} + \arg\det Y_d Y_u, \tag{2}$$

where θ_{QCD} is the θ parameter of QCD, while Y_u and Y_d are up and down Yukawa matrices respectively. This parameter cannot be eliminated from the Lagrangian. The chiral transformation

$$q \mapsto e^{i\overline{\theta}\gamma_5/2} q, \tag{3}$$

eliminates the $G\widetilde{G}$ term due to the chiral anomaly, however explicit breaking of the axial symmetry by the quark masses yields an extra contribution to the quark mass term:

$$\overline{q} m_q q \mapsto \overline{q} m_q \left(1 + e^{i\overline{\theta}\gamma_5}\right) q. \tag{4}$$

Additionaly, it can be shown that the electric dipole moment (EDM) of the neutron is proportional to the $\overline{\theta}$ parameter, thus we can determine the $\overline{\theta}$ parameter from a potential measurement of the neutron EDM. This, however is not been observed yet which yields the bound

$$|\overline{\theta}| \lesssim 10^{-10}. \tag{5}$$

The smallness of this parameter is known as the Strong CP Problem.

In this lecture, we will briefly list possible solutions to this problem, while concentrating mainly on the by far the most popular one; namely the axion. We will discuss the axions is a general setting before studying specific UV completions in the next lecture.

1 Non-axion solutions to the Strong CP problem

Before discussing the axion solution to the Strong CP problem extensively in the next section, let us start by mentioning alternative solutions.

Massless up quark

Remember that the reason why the $\bar{\theta}$ term cannot be eliminated by a chiral rotation of the form (3) is the fact that quark masses breaks the axial symmetry explicitly. Therefore, if only one of the quarks were massless, then the $\bar{\theta}$ term would be eliminated by applying the chiral rotation in Eq. (3) to the massless quark. For some time, this was considered a serious possibility because of the difficulties in extracting m_u/m_d from Chiral Perturbation Theory due to the Kaplan-Manohar ambiguity.[1] This possibility has now been ruled out by lattice simulations[2].[3]

[1] David B. Kaplan and Aneesh V. Manohar (May 1986). In: *Phys. Rev. Lett.* 56 (19), pp. 2004–2007.

[2] R. L. Workman et al. (2022). In: *PTEP* 2022, p. 083C01.

[3] Constantia Alexandrou et al. (2020). In: *Phys. Rev. Lett.* 125.23, p. 232001. arXiv: 2002.07802 [hep-lat].

Nelson-Bar mechanism

We know that the Standard Model is not valid up to arbitrary high energy scales, and it needs to be extended to a new model at high energies. It is plausible that this extended version has the CP symmetry built-in so the CP conservation at the QCD is just a result of this CP symmetry. This construction is known as the Nelson-Barr mechanism[4].[5] However, we know that the CP is broken in the Standard Model in weak interactions, so the challenge in constructing these models is to generate the correct chiral structure and the CKM phase while at the same time protect the strong sector from the breaking effects[6].[7]

[4] Ann Nelson (1984). In: *Physics Letters B* 136.5, pp. 387–391. ISSN: 0370-2693.

[5] S. M. Barr (July 1984). In: *Phys. Rev. Lett.* 53 (4), pp. 329–332.

[6] Luca Vecchi (2017). In: *JHEP* 04, p. 149. arXiv: 1412.3805 [hep-ph].

[7] Michael Dine and Patrick Draper (2015). In: *JHEP* 08, p. 132. arXiv: 1506.05433 [hep-ph].

Solutions within QCD

There is also the possibility that the Strong CP problem is solved within the QCD itself. It might be that the confinement can *screen* the effects of the $\bar{\theta}$ term so that the QCD preserves CP invariance.[8] However, this fails to provide a simultaneous solution to the $\mathrm{U}(1)_A$ problem.[9]

[8] Gregory Gabadadze and M. Shifman (2002). In: *Int. J. Mod. Phys. A* 17. Ed. by K. A. Olive, M. A. Shifman, and M. B. Voloshin, pp. 3689–3728. arXiv: hep-ph/0206123.

[9] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov (1980). In: *Nuclear Physics B* 166.3, pp. 493–506. ISSN: 0550-3213.

2 Axion solution to the Strong CP problem

It is fair to say that by a wide margin, the most popular solution to the Strong CP problem is using axions. To understand the essence of this solution, it is tremendously helpful to study the energy of the QCD vacuum state on the $\bar{\theta}$ parameter.

Vacuum energy of QCD

Let $E(\bar{\theta})$ denotes the energy of the QCD ground state with the parameter $\bar{\theta}$. This is related to the generating functional in the large Euclidean volume $\mathcal{V}_4$ limit as

$$Z(\bar{\theta}) = \lim_{\mathcal{V}_4\to\infty} e^{-E(\bar{\theta})\mathcal{V}_4}. \tag{6}$$

It is possible to calculate the vacuum energy by using semi-classical approximation and the result is

$$E(\bar{\theta}) = -2Ke^{-8\pi^2/g_s^2}\cos(\bar{\theta}), \tag{7}$$

where K is a positive factor which is calculable,[10] but it is not essential for the following discussion. The important point to see is that the energy is minimized at $\bar{\theta} = 0$, i.e. at the CP-conserving value! This fact can also be seen without relying on the semi-classical approximation by the following. Let $\mathcal{D}G := \mathcal{D}A\mathcal{D}q\mathcal{D}\bar{q}$ denotes the total QCD path integral measure. The Euclidean generating functional is [11]

$$\begin{aligned} Z(\bar{\theta}) &= \int \mathcal{D}G\, e^{-S_E} \\ &= \int \mathcal{D}G \exp\left\{-\frac{1}{4}\int \mathrm{d}^4x_E\, GG + \frac{i\bar{\theta}g_s^2}{32\pi^2}\int \mathrm{d}^4x_E\, G\widetilde{G}\right\} \\ &= \int \mathcal{D}G\, e^{-S_E(Q=0)+i\bar{\theta}Q}. \end{aligned} \tag{8}$$

We now use the fact that the path integral measure is positive definite for a vector-like theory[12] like QCD.[13] Then, the generating functional obeys

$$\begin{aligned} Z(\bar{\theta}) &= \left|\int \mathcal{D}G\, e^{-S_E(Q=0)+i\bar{\theta}Q}\right| \\ &\leq \int \mathcal{D}G \left|e^{-S_E(Q=0)+i\bar{\theta}Q}\right| \\ &= \int \mathcal{D}G \left|e^{-S_E(Q=0)}\right| \\ &= Z(0). \end{aligned} \tag{9}$$

So we have derived

$$Z(\bar{\theta}) \leq Z(0) \quad \Rightarrow \quad E(\bar{\theta}) \geq E(0). \tag{10}$$

This is the special case of the Vafa-Witten theorem which states that parity cannot be broken spontaneously in QCD.[14] This result is very important since it should apply even in the low energy theory where the result in Eq. (7) no longer holds due to the inapplicability of the semi-classical approximation. We will demonstrate this explicitly below, when we discuss the axion effective Lagrangian.

Strictly speaking, the Vafa-Witten theorem does not hold in the Standard Model. The reason is that the Standard Model is not a vector-like theory. It is chiral theory where the left-handed and right-handed fermions have different types of interactions. In fact, the $\bar{\theta}$ parameter

[10] Sidney Coleman (1985). "The uses of instantons". In: *Selected Erice lectures. Aspects of Symmetry*. Cambridge University Press.

[11] The extra i factor at the second step of Eq. (8) arises during the transition from the Minkowski space to the Euclidean space. See Lecture 6 for a reminder.

[12] By vector-like we mean that the left- and right-handed parts of the fermions are treated on equal footing. This is not the case, for example, in Standard Model where the weak interactions of left-handed fermions differ substantially from the right-handed ones, see Lecture 7. The consequences of this for the vacuum energy of QCD will be explained below.

[13] C. Vafa and E. Witten (1984a). In: *Nuclear Physics B* 234.1, pp. 173–188. ISSN: 0550-3213.

[14] Cumrun Vafa and Edward Witten (Aug. 1984b). In: *Phys. Rev. Lett.* 53 (6), pp. 535–536.

receive renormalization group corrections $\delta\overline{\theta}$ due to the existence of the CP-violating phase in the CKM matrix. However, it turns out that these corrections are heavily suppressed due to a non-trivial screening mechanism in the Standard Model[15,16], and they are well below the experimental sensitivity $\delta\overline{\theta} \ll 10^{-10}$.

[15] John Ellis and Mary K. Gaillard (1979). In: *Nuclear Physics B* 150, pp. 141–162. ISSN: 0550-3213.

[16] This screening mechanism is the main reason behind the challenge in constructing succesful Nelson-Barr models.

Making $\overline{\theta}$ dynamical

We have just seen that out of infinetely many QCD vacua parametrized by the $\overline{\theta}$ parameter, the one with $\overline{\theta} = 0$ has the lowest vacuum energy. This does not explain why $\overline{\theta} = 0$ though. The reason is that $\overline{\theta}$ is not *dynamical* in QCD. But, if $\overline{\theta}$ *were dynamical*, then the system dynamics will make it vanish since in that case the energy is minimized. This is the essence of the axion solution to the Strong CP problem. One simply promotes $\overline{\theta}$ to a dynamical field.

Let's see how can this be done explicitly. The simplest possibility is to use a scalar field which we denote as ϕ. The main requirement for the axion solution to work is that this scalar field has a quasi shift symmetry

$$\phi \mapsto \phi + \alpha f_\phi, \tag{11}$$

which leaves the action invariant up to an anomaly term:

$$\delta S = \frac{\alpha g_s^2}{32\pi^2} \int \mathrm{d}^4 x\, G\widetilde{G}. \tag{12}$$

In Eq. (11), α is a real parameter, and f_ϕ is a energy scale that we specify shortly. In order this requirement to be satisfied, the scalar field Lagrangian should have the term

$$\mathcal{L}_\phi \supset \frac{\phi}{f_\phi} \frac{g_s^2}{32\pi^2} G\widetilde{G}. \tag{13}$$

Then, we can make use of the shift symmetry in Eq. (11) by choosing α such that the $\overline{\theta}$ term is cancelled via the contribution in Eq. (12). The Vafa-Witten theorem that we have just proved ensures that the scalar field is minimized at the CP-conserving value, i.e. $\langle\phi\rangle = 0$, in any vector-like theory. Thus, the Strong CP problem is solved.

Including the kinetic term, the most general Lagrangian for ϕ reads

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 + \frac{\phi}{f_\phi} \frac{g_s^2}{32\pi^2} G\widetilde{G} + \mathcal{L}_{\text{int}}(\partial_\mu \phi, \ldots), \tag{14}$$

where $(\partial_\phi)^2 := (\partial_\mu\phi)(\partial^\mu\phi)$ and $\mathcal{L}_{\text{int}}$ describes interactions other than the $G\widetilde{G}$ interaction. Note that the latter depends only on the derivatives of the axion due to the shift symmetry.

This Lagrangian is not-renormalizable due to the existence of a dimensionful coupling so one needs a UV completion. If we remember our very first lecture, Lecture 1b, it is not hard to come up with a possible UV completion. The shift symmetry and the derivative interactions are precisely properties of the Goldstone bosons that arise from the spontaneous breaking of continuous global symmetries. The additional requirement that the action is invariant up to the term given in Eq. (12) implies that the global symmetry should be anomalous at least under the QCD. The simplest choice for this global symmetry is of course a $\mathsf{U}(1)$. When this symmetry is spontaneously broken around the energy scale f_ϕ, there should be two degrees of freedom in the broken phase. One is the radial mode which is heavy $m_r \sim f_\phi$, while the other is the angular mode which we identify as ϕ and call it as the axion. Shortly, we will see that the $\phi G\widetilde{G}$ term introduces a potential for the axion, thereby gives it a non-zero mass. Therefore, the axion is a pseudo-Nambu-Goldstone boson (pNGB). The symmetry breaking scale f_ϕ is known as the axion decay constant.

Historically, the existence of the global $\mathsf{U}(1)$ symmetry is first postulated by Roberto Daniele Peccei and Helen Quinn[17].[18] For this reason, the $\mathsf{U}(1)$ symmetry is commonly referred as the Peccei-Quinn symmetry and denoted by $\mathsf{U}(1)_{\mathrm{PQ}}$. The existence of the light degree of freedom, the axion, is discovered by Frank Wilczek[19] and Steven Weinberg.[20] The name "axion" is given by Wilczek who is inspired from a laundry detergent, see Figure 1.

Figure 1: A picture of the laundry detergent where the name "axion" is inspired. Cropped from the picture Axions are named after a laundry detergent! by Marc Buehler that is licensed under CC BY-NC 2.0.

[17] R. D. Peccei and Helen R. Quinn (1977b). In: *Phys. Rev. Lett.* 38, pp. 1440–1443.

[18] R. D. Peccei and Helen R. Quinn (Sept. 1977a). In: *Phys. Rev. D* 16 (6), pp. 1791–1797.

[19] Frank Wilczek (1978). In: *Phys. Rev. Lett.* 40, pp. 279–282.

[20] Steven Weinberg (1978). In: *Phys. Rev. Lett.* 40, pp. 223–226.

3 Chiral Lagrangian with Axions

We see that an axion solves the Strong CP problem if it has a low-energy coupling to QCD given by $\phi G\widetilde{G}$. Therefore, axion is present also in the Chiral Lagrangian. In this section, we will construct the Chiral Lagrangian including axions. This calculation will also give us the effective low-energy potential of the axion, and hence the axion mass.

For simplicity we consider the two-flavor QCD with the definitions

$$\begin{aligned} q &= \begin{pmatrix} u \\ d \end{pmatrix}, \quad \overline{q} = \begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix}, \\ q_{L,R} &= \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}, \quad \overline{q}_{L,R} = \begin{pmatrix} u^\dagger_{L,R} & d^\dagger_{L,R} \end{pmatrix}. \end{aligned} \tag{15}$$

We consider the effective Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\phi}{f_\phi}\frac{g_s^2}{32\pi^2}G\widetilde{G} + \frac{1}{4}g^0_{\phi\gamma}\phi F\widetilde{F} + \frac{\partial_\mu\phi}{2f_\phi}\overline{q}c^0_q\gamma^\mu\gamma_5 q - \overline{q}M_q q, \tag{16}$$

where $M_q = \mathrm{diag}(m_u, m_d)$ is the diagonalized quark mass matrix. We have also introduced two model-dependent couplings $g^0_{\phi\gamma}$ and

$c_q^0 = \mathrm{diag}(c_u^0, c_d^0)$ whose motivation will be clear shortly. The $\phi G\widetilde{G}$ term creates a coupling between the axion and the gluons. It is more convenient to perform a field-dependent chiral rotation of quarks to eliminate this term. We consider the following transformation:

$$q \mapsto \exp\left\{i\gamma_5 \frac{\phi}{2f_\phi} U_\phi\right\} q, \tag{17}$$

where U_ϕ is a generic 2×2 matrix acting on the quark fields. Without loss of generality, we can choose it to be a diagonal matrix [21]. Due to the chiral anomaly, this generates the term

[21] For non-diagonal U_ϕ, we can first go to a basis where U_ϕ acts on the quark field diagonally. Such a transformation is not anomalous, and will not generate additional terms.

$$-\left(\mathrm{Tr}\, U_\phi\right) \frac{\phi}{f_\phi} \frac{g_s^2}{32\pi^2} G\widetilde{G}. \tag{18}$$

By choosing U_ϕ such that $\mathrm{Tr}\, U_\phi = 1$, we can cancel the axion-gluon term. However, such a transformation will also modifies other terms in effective Lagrangian in Eq. (16):

- The mass matrix M_q is modified to

$$M_q \mapsto M_\phi := \exp\left\{i\frac{\phi}{2f_\phi} U_\phi\right\} M_q \exp\left\{i\frac{\phi}{2f_\phi} U_\phi\right\}. \tag{19}$$

- It modifies the coupling between the axion and the axial quark current:[22]

$$c_q^0 \mapsto c_q := c_q^0 - U_\phi. \tag{20}$$

[22] To see this remember that under a field-dependent axial transformation $\psi \mapsto e^{i\epsilon(x)\gamma_5}\psi$, the action is transformed as

$$\delta S = \int \mathrm{d}^4x \left(\partial_\mu \epsilon(x)\right) J_5^\mu$$

where J_5^μ is the axial current:

$$J_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi.$$

See Lecture 4 if you need to refresh your memory. The result in Eq. (20) is a straightforward generalization of these results.

- Finally, this transformation is also anomalous under QED since quarks have electric charges. The term generated by the anomaly modifies the axion-photon coupling as [23]

$$g_{\phi\gamma}^0 \mapsto g_{\phi\gamma} := g_{\phi\gamma}^0 - (2N_c)\frac{\alpha_{\mathrm{EM}}}{2f_\phi} \mathrm{Tr}\left(U_\phi Q_{\mathrm{EM}}^2\right), \tag{21}$$

  where $N_c = 3$ is the number of colors, $\alpha_{\mathrm{EM}} = e^2/4\pi$ is the fine-structure constant, and Q_{EM} is the diagonal matrix with the electric charges of the up and down quarks respectively,

$$Q_{\mathrm{EM}} = \mathrm{diag}(2/3, -1/3). \tag{22}$$

[23] This is again a straightforward generalization of the results that we have derived in Lecture 4. The factor N_c comes from the fact a chiral rotation to a single quark rotates N_c numbers of color degrees of freedom.

After all these modifications, we can write the axion effective Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{4} g_{\phi\gamma}\phi F\widetilde{F} + \frac{\partial_\mu\phi}{2f_\phi}\overline{q} c_q \gamma^\mu\gamma_5 q - \overline{q} M_\phi q. \tag{23}$$

Our next task is to derive various axion properties from this. Since we are interested in low-energy properties, Chiral Lagrangian is useful for this task.

Axion potential and axion mass

In Lecture 3, we have derived an expression for the mass term in Chiral Lagrangian:

$$\mathcal{L}_{\chi,\text{mass}} = \frac{\mu}{4} f_\pi^2 \left[\text{Tr}\left(\Sigma^\dagger M \right) + \text{Tr}\left(M^\dagger \Sigma \right) \right], \tag{24}$$

where for two-flavor QCD, Σ reads

$$\Sigma = \exp\left\{ 2i \frac{\pi^a}{f_\pi} \tau^a \right\} = \exp\left\{ \frac{i}{f_\pi} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix} \right\}. \tag{25}$$

For the calculation it is useful to know the identity:

$$\Sigma = 1 \cos \frac{\pi}{f_\pi} + \frac{i\sigma^a \pi^a}{\pi} \sin \frac{\pi}{f_\pi}, \tag{26}$$

where 1 is the 2×2 identity matrix, and we have defined

$$\pi := \sqrt{(\pi^0)^2 + 2\pi^- \pi^+}. \tag{27}$$

Then setting $M = M_\phi$ in Eq. (24) where M_ϕ is given by Eq. (19) yields to a combined potential for the axion and the pions:

$$\begin{aligned} V(\phi, \pi^a) = -\frac{m_\pi^2 f_\pi^2}{2(m_u + m_d)} \Bigg\{ & \left[m_u \cos\left(U_\phi^u \frac{\phi}{f_\phi} \right) + m_d \cos\left(U_\phi^d \frac{\phi}{f_\phi} \right) \right] \cos\left(\frac{\pi}{f_\pi} \right) \\ & + \frac{\pi^0}{\pi} \left[m_u \sin\left(U_\phi^u \frac{\phi}{f_\phi} \right) - m_d \sin\left(U_\phi^d \frac{\phi}{f_\phi} \right) \right] \sin\left(\frac{\pi}{f_\pi} \right) \Bigg\}, \end{aligned} \tag{28}$$

where U_ϕ^u and U_ϕ^d are the diagonal components of U_ϕ, and we have replaced the μ with the pion mass via

$$m_\pi^2 = \frac{\mu}{2}(m_u + m_d). \tag{29}$$

For generic values of U_ϕ^u and U_ϕ^d, the potential in Eq. (28) contains a mass mixing between the axion and the neutral pion π^0. This mixing can be removed by choosing U_ϕ^u and U_ϕ^d such that $U_\phi^u m_u - U_\phi^d m_d = 0$. This condition together with $\text{Tr}\, U_\phi = 1$ fixes these terms to be

$$U_\phi^u = \frac{m_d}{m_u + m_d} \quad \text{and} \quad U_\phi^d = \frac{m_u}{m_u + m_d}. \tag{30}$$

Moreover, it is easy to check the potential in Eq. (28) when all the pions fields vanish which implies all the pions have vanishing VEV as expected. Then, on the pion ground state $\pi^a = 0$ we can substitute Eq. (30) into Eq. (28) and expand the potential around $\phi/f_\phi = 0$ to get

$$V(\phi) := V(\phi, \pi^a = 0) = -m_\pi^2 f_\pi^2 + \frac{1}{2} \frac{m_\pi^2 f_\pi^2}{f_\phi^2} \frac{m_u m_d}{(m_u + m_d)^2} \phi^2 + \mathcal{O}\left(\left(\frac{\phi}{f_\phi} \right)^4 \right). \tag{31}$$

From this, we can directly read off the axion mass as

$$m_\phi = \frac{m_\pi f_\pi}{f_\phi} \frac{\sqrt{m_u m_d}}{m_u + m_d} \simeq 5.7 \left(\frac{10^{12}\,\mathrm{GeV}}{f_\phi} \right) \mu\mathrm{eV}. \tag{32}$$

We see that the QCD scale $\Lambda_{\rm QCD} \sim \sqrt{m_\pi f_\pi}$ determines a key relation between the axion mass and the axion decay constant. As we shall see at the end of this lecture, this is the property that distinguishes between the QCD axion and an Axion-Like-Particle (ALP).

Even though the choice of the matrix U_ϕ according to Eq. (30) is useful to obtain the axion mass, it is not convenient to describe the properties of the axion-pion potential away from the minimum. For the latter, a more convenient choice is

$$U_\phi = \mathrm{diag}\left(\frac{1}{2}, \frac{1}{2} \right). \tag{33}$$

Plugging this into Eq. (28) and setting $\pi^\pm = 0$ yields a potential for the axion and the neutral pion:[24]

$$V(\phi, \pi^0) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2\left(\frac{\phi}{2 f_\phi} \right)} \cos\left(\frac{\pi^0}{f_\pi} - \varphi_\phi \right), \tag{34}$$

where

$$\tan \varphi_\phi := \frac{m_u - m_d}{m_u + m_d} \tan\left(\frac{\phi}{2 f_\phi} \right). \tag{35}$$

This form of the potential makes is manifest that the global minimum is at $(\phi, \pi^0) = (0, 0)$, and there is a potential valley at $\pi^a = \varphi_\phi f_\pi$. At this valley, the axion potential reads

$$V(\phi) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2\left(\frac{\phi}{2 f_\phi} \right)}. \tag{36}$$

By expanding this potential around $\phi = 0$, one can check that it gives the same axion mass as in Eq. (32), and also the coefficients of the $\mathcal{O}(\phi^4)$ terms agree precisely.

[24] Giovanni Grilli di Cortona et al. (2016). In: *JHEP* 01, p. 034. arXiv: `1511.02867 [hep-ph]`.

Axion-pion coupling

To obtain the couplings between the axion and the pions, we need to consider the axial quark current which can conveniently be decomposed as

$$\bar{q}\gamma^\mu\gamma_5 q = \underbrace{\frac{1}{2}\mathrm{Tr}(c_q)\bar{q}\gamma^\mu\gamma_5 q}_{\text{iso-singlet}} + \underbrace{\frac{1}{2}\mathrm{Tr}(c_q\sigma^a)\bar{q}\gamma^\mu\gamma_5\sigma^a q}_{\text{iso-triplet}}, \tag{37}$$

where we have used the identity

$$(\sigma^a)_{ij}(\sigma^a)_{kl} = 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}. \tag{38}$$

The prefix "iso-" stands for the isospin which is the residual $\mathsf{SU}(2)_V$ symmetry that survives the chiral symmetry breaking

$$\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R \to \mathsf{SU}(2)_V. \tag{39}$$

The terms singlet and triplet denotes how the corresponding terms transform under the isospin. The first term transforms as a scalar, hence it is a singlet, while the second term transforms as a three-dimensional vector, hence it is a triplet. Under the isospin, pions $\pi^0, \pi^\pm$ form a triplet, so to obtain the coupling between the axions and the pions it suffices to consider only the iso-triplet term in Eq. (37).

Now we need to write the Chiral Lagrangian including the term that corresponds to this iso-triplet term. Including the kinetic term for the pions, the chiral Lagrangian now reads[25]

$$\mathcal{L}_\chi = \frac{f_\pi^2}{4} \operatorname{Tr}\left[\left(\partial^\mu \Sigma^\dagger\right)\left(\partial_\mu \Sigma\right)\right] + \mathcal{L}_{\chi,\text{mass}} + \frac{\partial^\mu \phi}{2 f_\phi}\frac{1}{2}\operatorname{Tr}\left(c_q \sigma^a\right) J_a^\mu, \tag{40}$$

[25] In general, the regular derivatives in the kinetic term of Eq. (40) should be replaced with covariant derivatives such that

$$\partial_\mu \Sigma \to D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu[Q,\Sigma],$$

where $Q = \operatorname{diag}(2/3, -1/3)$ is the quark electric charge matrix, in order to derive electromagnetic couplings of the pions. We are omitting these since they are not relevant to our discussion of the axion couplings.

where $\mathcal{L}_{\chi,\text{mass}}$ is given by (24), and J_a^μ is yet to be determined. Note that the $\bar{q}\gamma^\mu\gamma_5\sigma^a q$ is nothing but the Noether current that arise from an $\mathsf{SU}(2)$ axial transformation to quarks[26]. In the chiral Lagrangian, the term J_a^μ should play the same role. Thus, it can be determined from the axial $\mathsf{SU}(2)$ current derived from the Lagrangian in Eq. (40). The result is

$$J_a^\mu = \frac{i}{2} f_\pi^2 \operatorname{Tr}\left[\sigma^a\left(\Sigma\partial_\mu\Sigma^\dagger - \Sigma^\dagger\partial_\mu\Sigma\right)\right]. \tag{41}$$

[26] To show this remember that a generic $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$ transformation to quarks can be represented as

$$\exp\{i\alpha^a\sigma^a + \gamma_5\beta^a\sigma^a\}$$

where the axial transformations form a subset where $\alpha^a = 0$. See Lecture 3 for a reminder. Then, the Noether current can be derived by using the quark kinetic term $i\bar{q}\not{D}q$.

This is called the pion iso-triplet axial-vector current. Expanding it yields

$$\begin{aligned}\frac{\partial_\mu\phi}{2f_\phi}\frac{1}{2}\operatorname{Tr}\left(c_q\sigma^a\right)J_a^\mu \simeq &-\frac{1}{2}\left(\frac{m_d-m_u}{m_u+m_d}+c_d^0-c_u^0\right)\frac{f_\pi}{f_\phi}\partial_\mu\phi\partial^\mu\pi^0 \\ &+\frac{1}{3}\left(\frac{m_d-m_u}{m_u+m_d}+c_d^0-c_u^0\right)\frac{\partial_\mu\phi}{f_\pi f_\phi}\left(2\partial^\mu\pi^0\pi^+\pi^- - \pi^0\partial^\mu\pi^+\pi^- - \pi^0\pi^+\partial^\mu\pi^-\right)\end{aligned} \tag{42}$$

The first term introduces a kinetic mixing of the order of

$$\epsilon = -\frac{1}{2}\left(\frac{m_d-m_u}{m_d+m_u}+c_d^0-c_u^0\right)\frac{f_\pi}{f_\phi} \ll 1 \tag{43}$$

which can be neglected. The second term gives the axion-pion coupling:

$$C_{\phi\pi} = -\frac{1}{3}\left(c_u^0 - c_d^0 - \frac{m_d-m_u}{m_u+m_d}\right). \tag{44}$$

Axion-Photon coupling

Using the choice

$$U_\phi^u = \frac{m_d}{m_u + m_d} \quad , \quad U_\phi^d = \frac{m_u}{m_u + m_d} \tag{45}$$

to ensure no axion-pion mass mixing we find the axion-photon coupling as

$$\begin{aligned} g_{\phi\gamma} &= g_{\phi\gamma}^0 - (2N_c)\frac{\alpha_{\text{em}}}{2\pi f_\phi}\,\text{Tr}\left(U_\phi Q_{\text{em}}^2\right) \\ &= g_{\phi\gamma}^0 - \frac{\alpha_{\text{em}}}{2\pi f_\phi}\left(\frac{2}{3}\frac{4m_d + m_u}{m_u + m_d}\right). \end{aligned} \tag{46}$$

Axion-nucleon coupling

At low energy one can introduce nucleon as a iso-spin doublet:

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{47}$$

where p and n stands for the proton and the neutron respectively. The axion-nucleon coupling can be defined in an analogous way to the axion-quark current:[27]

[27] Grilli di Cortona et al., "The QCD axion, precisely".

$$\frac{\partial_\mu \phi}{2f_\phi}\overline{N}C_{\phi N}\gamma^\mu\gamma_5 N \quad , \quad C_{\phi N} = \text{diag}\left(C_{\phi p}, C_{\phi n}\right), \tag{48}$$

where

$$C_{\phi p} = c_u^0 \Delta u + c_d^0 \Delta d - \left(\frac{m_d}{m_u + m_d}\Delta u + \frac{m_u}{m_u + m_d}\Delta d\right), \tag{49}$$

$$C_{\phi n} = c_d^0 \Delta u + c_u^0 \Delta d - \left(\frac{m_u}{m_u + m_d}\Delta u + \frac{m_d}{m_u + m_d}\Delta d\right). \tag{50}$$