Lecture 7: Strong CP Problem

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In the lectures so far, we have derived two important non-trivial results:

1. The non-trivial vacuum structure of QCD requires us to add an additional term to the QCD Lagrangian which is the θ-term:
$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \frac{\theta g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}. \tag{1}$$

2. Under a chiral transformation with a single quark $q \mapsto e^{i\alpha\gamma_5} q$, the path fermion path integral measure is not invariant, and as a result the QCD Lagrangian transforms as, see Lecture 4,
$$\mathcal{L}_{\text{QCD}} \mapsto \mathcal{L}_{\text{QCD}} - \frac{\alpha g_s^2}{16\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}. \tag{2}$$

In this lecture we will combine these two results and state precisely the Strong CP problem.

1 Quark masses in the Standard Model

Electroweak symmetry breaking

Recall that the Standard Model (SM) is based on the gauge group

$$\mathsf{SU}(3)_C \otimes \mathsf{SU}(2)_{EW} \otimes \mathsf{U}(1)_Y,$$

where C, EW and Y stands for color, electroweak, and hypercharge respectively. The fermion masses are generated via the Electroweak symmetry breaking where the subgroup $\mathsf{SU}(2)_{EW} \otimes \mathsf{U}(1)_Y$ is spontaneously broken to $\mathsf{U}(1)_{EM}$ where the latter stands for the usual electromagnetism. Note that this is a phenomenon where a *gauge (local) symmetry* is spontaneously broken which is quite different to the phenomena that we have studied in the beginning of the course where the broken symmetries were *global*.

The gauge group $\mathsf{SU}(2)_{EW} \otimes \mathsf{U}(1)_Y$ describes the Electroweak Theory which unifies the weak interactions and the electromagnetism. The $\mathsf{SU}(2)$ part consists of three gauge bosons $\left\{W^a_\mu\right\}_{a=1}^3$ where $\mathsf{U}(1)_Y$ has B_μ. For the electroweak symmetry breaking to occur, one also needs the Higgs multiplet H which is a complex doublet that transforms under the fundamental of $\mathsf{SU}(2)_{EW}$ and has hypercharge $+\frac{1}{2}$ [1]. The Lagrangian is

$$\mathcal{L}_{\rm EW} = -\frac{1}{4}W^a_{\mu\nu}W^{a\,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D^\mu H)^\dagger(D_\mu H) - V(H^\dagger H), \qquad (3)$$

where $W^a_{\mu\nu}$ and $B_{\mu\nu}$ are field strengths for the $\mathsf{SU}(2)_{EW}$ and $\mathsf{U}(1)_Y$ gauge bosons respectively. The covariant derivative is given by[2]

$$D_\mu H = \left(\partial_\mu - igW^a_\mu \tau^a - \frac{1}{2}g' B_\mu\right)H, \qquad (4)$$

where g and g' are the gauge couplings for the $\mathsf{SU}(2)_{EW}$ and $\mathsf{U}(1)_Y$ respectively, $\{\tau^a\}$ are the $\mathsf{SU}(2)$ generators[3], and the coefficient in front of the B_μ term comes from the hypercharge of the Higgs doublet.

The potential $V(H^\dagger H)$ is such that the Higgs doublet obtains a VEV which without loss of generality can be chosen as

$$\langle H \rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (5)$$

where $v \simeq 250\,\mathrm{GeV}$. Like we have done in the global symmetry breaking case, we can write the Higgs doublet around this VEV as

$$H = \frac{1}{\sqrt{2}}\exp\left\{2i\frac{\pi^a\tau^a}{v}\right\}\begin{pmatrix} 0 \\ v+h \end{pmatrix}, \qquad (6)$$

where h is a scalar excitation which will be identified with the Higgs particle. Remember that in the case of spontaneous breaking of a global symmetry, $\{\pi^a\}$ were the Goldstone bosons. The transformation law of the scalar field that gets a VEV under the broken symmetry determine the transformation of the Goldstone bosons. The situation here is very different. The broken symmetry is a gauge symmetry. So the resulting transformation law for $\{\pi^a\}$ describes a gauge redundancy[4] We can make use of this gauge freedom to set $\pi^a = 0$. This is called the Unitary gauge. This choice is very convenient in the study of spontaneous breaking of gauge symmetries since it removes the kinetic mixing between π's and the gauge bosons.

Using Eqs. (4) and (6) with $\pi^a = 0$ one finds

$$\begin{aligned}(D^\mu H)^\dagger(D_\mu H) = \frac{g^2v^2}{8}\bigg[&W^1_\mu W^{1\,\mu} + W^2_\mu W^{2\,\mu} \\ &+\left(\frac{g'}{g}B^\mu - W^{3\,\mu}\right)\left(\frac{g'}{g}B_\mu - W^3_\mu\right) \\ &+ h \text{ terms}\bigg],\end{aligned} \qquad (7)$$

where we have neglected the h-dependent terms for now. This result shows that the gauge bosons W^3_μ and B_μ mix with each other. To find

[1] Hypercharge is the quantum number corresponding to the $\mathsf{U}(1)_Y$ gauge symmetry.

[2] In Eq. (4), $\left\{W^a_\mu\right\}$ and B_μ are the gauge bosons of $\mathsf{SU}(2)_{EW}$ and $U(1)_Y$ respectively.

[3] Recall that these can be written in terms of Pauli matrices as $\tau^a = \sigma^a/2$.

[4] This is in contrast to the previous case where the transformation law dictated the physical properties of the Goldstone bosons, i.e. the fact they are massless, and have derivative couplings.

the spectrum we need to diagonalize the mass terms. This can be achieved by defining

$$Z_\mu := \cos\theta_w W^3_\mu - \sin\theta_w B_\mu \quad \text{and} \quad A_\mu := \sin\theta_w W^3_\mu + \cos\theta_w B_\mu, \tag{8}$$

with

$$\tan\theta_w := \frac{g'}{g}. \tag{9}$$

so that

$$B_\mu = \cos\theta_w A_\mu - \sin\theta_w Z_\mu \quad \text{and} \quad W^3_\mu = \sin\theta_w A_\mu + \cos\theta_w Z_\mu. \tag{10}$$

Then, with a little bit of algebra one can show that the Lagrangian in Eq. (3) contains terms like

$$\mathcal{L}_{\text{EW}} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_Z^2 Z^\mu Z_\mu, \quad m_Z = \frac{gv}{2\cos\theta_w}, \tag{11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$. Here, A_μ is the photon of electromagnetism, while Z_μ is a massive spin-1 boson known as the Z-boson. From Eq. (7) it is easy to see that the electroweak bosons W^1_μ and W^2_μ also become massive. It is more convenient to define

$$W^\pm_\mu := \frac{1}{\sqrt{2}}\left(W^1_\mu \mp W^2_\mu\right), \tag{12}$$

where W^+_μ and W^-_μ refer to the eigenstates which are charged positively and negatively under the electromagnetism. They contribute to the Lagrangian via

$$\mathcal{L} \supset -\frac{1}{2}\left(\partial_\mu W^+_\nu - \partial_\nu W^-_\mu\right)\left(\partial^\mu W^{+\,\nu} - \partial^\nu W^{-\,\mu}\right) + m_W^2 W^+_\mu W^{-\,\mu}. \tag{13}$$

These are the W bosons of the Standard Model.

We see that as a result of spontaneous symmetry breaking, three of the four gauge bosons acquire masses. A fundamental result of the Quantum Field Theory states that the massless particles have two degrees of freedom independent of their spins, while the number of degrees of freedom in massive particles is $2s+1$ where s is the spin. So, when the gauge symmetry is spontaneously broken, the W and the Z bosons should acquire an additional degree of freedom. This comes from the complex Higgs doublet which had four degrees of freedom before the symmetry breaking. Its three degrees of freedom are transferred to the gauge bosons while the remaining degree of freedom is the scalar excitation h in Eq. (6) which is nothing but the Higgs boson of the Standard Model. A common way to explain this phenomenon is to say that the gauge bosons eat the degrees of freedom of the Higgs doublet to become massive.

Fermions in the Standard Model

The theory of electroweak interactions is chiral and maximally parity violating since the $\mathsf{SU}(2)_{EW}$ gauge bosons only couple to the left-handed fermions. This is just an observational fact which cannot be explained in the Standard Model.

The above statement means that the left-handed fermions should form doublets that transform under the fundamental representation of $\mathsf{SU}(2)_{EW}$. Among these are the lepton doublets

$$L_i = \left\{ \begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix} \right\}, \tag{14}$$

where e, μ, τ stands for electron, muon, and tau respectively, and ν's are the corresponding neutrinos. The superscript i labels the generation. We also have the quark doublets:

$$Q_i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}. \tag{15}$$

On the other hand, the right-handed fermions are singlets under $\mathsf{SU}(2)_{EW}$ so they are represented by

$$e_R^i = \{e_R, \mu_R, \tau_R\}, \quad u_R^i = \{u_R, c_R, t_R\}, \quad d_R^i = \{d_R, s_R, b_R\}. \tag{16}$$

One needs to specify the hypercharges. These are given by

$$\begin{aligned} &L : -\frac{1}{2}, \qquad e_R : -1, \qquad Q : \frac{1}{6} \\ &u_R : \frac{2}{3}, \qquad d_R : -\frac{1}{3}, \qquad H : \frac{1}{2} \end{aligned} \tag{17}$$

Fermion masses

Now recall that a Dirac mass term for a fermion can be written as

$$\mathcal{L} = m\Big(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\Big). \tag{18}$$

However, we cannot write this term explicitly since it breaks the $\mathsf{SU}(2)_{EW}$ gauge invariance. But, we can write a term like

$$\mathcal{L}_{\text{yuk}} = -y\overline{L}He_R + \text{h.c.}, \tag{19}$$

where $\overline{L} := \begin{pmatrix} \nu_{e_L}^\dagger & e_L^\dagger \end{pmatrix}$, H is the Higgs doublet and y is a dimensionless coupling. After the electroweak symmetry breaking, H gets a VEV and this term becomes

$$\mathcal{L}_{\text{yuk}} \to -y\frac{v}{\sqrt{2}}\Big(e_L^\dagger e_R + e_R^\dagger e_L\Big). \tag{20}$$

This is a mass term for the electron where $m_e = yv/\sqrt{2}$. An interaction of the form given in Eq. (19) is called a Yukawa interaction, and y is called the Yukawa coupling. With terms like these, the charged leptons electron, muon, tau, and the down-type quarks d, s, b get their masses.

In order to give up-type quarks u, c and t their masses, one uses an interaction of the form[5]

$$\mathcal{L} \supset -y\overline{Q}\widetilde{H}u_R, \quad \text{where} \quad \widetilde{H} := i\sigma^2 H^*. \tag{21}$$

[5] To show that this term is $\mathsf{SU}(2)_{EW}$ invariant one needs to remember that both L and H transform under the fundamental representation of $\mathsf{SU}(2)_{EW}$, and invoke the identity

$$\sigma_j^\intercal \sigma_2 + \sigma_2\sigma_j = 0.$$

Therefore, all the quark masses can be generated via the Yukawa interaction

$$\begin{aligned}\mathcal{L}_{\text{yuk}} &= -Y^d_{ij}\overline{Q}^i H d^j_R - Y^u_{ij}\overline{Q}^i\widetilde{H}u^j_R + \text{h.c.} \\ &\to -\frac{v}{\sqrt{2}}\left(Y^d_{ij}\overline{Q}^i\begin{pmatrix}0\\1\end{pmatrix}d^j_R + Y^u_{ij}\overline{Q}^i(i\sigma^2)\begin{pmatrix}0\\1\end{pmatrix}u^j_R + \text{h.c.}\right)\end{aligned} \tag{22}$$

where there is implicit summation over the generation indices i and j. By introducing vectors

$$\mathbf{u}_L = \{u_L, c_L, t_L\}, \tag{23}$$

$$\mathbf{d}_L = \{d_L, s_L, b_L\}, \tag{24}$$

$$\mathbf{u}_R = \{u_R, c_R, t_R\}, \tag{25}$$

$$\mathbf{d}_R = \{d_R, s_R, b_R\}, \tag{26}$$

we can write Eq. (22) as

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}\left(\mathbf{d}_L^\dagger\, Y_d\, \mathbf{d}_R + \mathbf{u}_L^\dagger\, Y_u\, \mathbf{u}_R\right) + \text{h.c.}\,, \tag{27}$$

where Y_u and Y_d are called up and down Yukawa matrices respectively. Note that these matrices are not diagonal in general which implies that the mass matrix is also not diagonal. The expression in Eq. (27) is written in the flavour basis since it is written in terms of quark flavours u, d, s, c, b, and t.

It is possible to diagonalize the mass matrix though. In general, the Yukawa matrices are not hermitian. However, $Y_d Y_d^\dagger$ and $Y_u Y_u^\dagger$ are. So as a consequence of the finite dimensonal spectral theorem they can be diagonalized via the unitary matrices, and the resulting diagonal matrix has only real entries. So we can write

$$Y_d Y_d^\dagger = U_d M_d^2 U_d^\dagger \quad \text{and} \quad Y_u Y_u^\dagger = U_u M_u^2 U_u^\dagger, \tag{28}$$

where U_u and U_d are unitary, M_u and M_d are diagonal matrices. With these, we can write the Yukawa matrices as

$$Y_d = U_d M_d K_d^\dagger \quad \text{and} \quad Y_u = U_u M_u K_u^\dagger, \tag{29}$$

where K_u and K_d are again unitary matrices. With these definitions, the mass Lagrangian in Eq. (27) takes the form

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}\left(\mathbf{d}_L^\dagger U_d M_d K_d^\dagger \mathbf{d}_R + \mathbf{u}_L^\dagger U_u M_u K_u^\dagger \mathbf{u}_R\right) + \text{h.c.}\,. \tag{30}$$

This is still in flavour basis. To get a Lagrangian in the so-called mass basis, we make a change of basis via

$$\mathbf{d}_R \mapsto K_d \mathbf{d}_R, \quad \mathbf{u}_R \mapsto K_u \mathbf{d}_u, \quad \mathbf{d}_L \mapsto U_d \mathbf{d}_L, \quad \mathbf{u}_L \mapsto U_u u_L. \tag{31}$$

In this basis, the mass matrix becomes diagonal:

$$\mathcal{L}_{\text{mass}} = -\left(m_i^d d_{L,i}^\dagger d_{R,i} + m_i^u u_{L,i}^\dagger u_{R,i}\right) + \text{h.c.}\,, \tag{32}$$

where $\left\{m_i^d\right\}$ and $\{m_i^u\}$ are the elements of the diagonal matrices $vM_d/\sqrt{2}$ and $vM_u/\sqrt{2}$ respectively.

Cabibbo-Kobayashi-Maskawa (CKM) matrix

The interactions of the quarks with the electroweak gauge bosons are flavour-diagonal which means that the gauge interactions do not mix the flavours. However, by performing the change of basis given in Eq. (31), these couplings will also be modified. It turns out that only charged boson couplings are modified by this change of basis, and the modifications is encoded in the Cabibbo-Kobayashi-Maskawa (CKM) matrix:

$$V := U_u^\dagger U_d = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} =: \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{33}$$

This matrix is a 3×3 complex unitary matrix, so it has 9 free parameters, 3 angles and 6 phases. However, many of these parameters can be eliminated by noting that there is still a $\mathsf{U}(1)^6$ global symmetry which corresponds to separate $\mathsf{U}(1)_V$ rotations for each of the six quarks:

$$\begin{aligned} d_L^j &\mapsto e^{i\alpha_j} d_L^j \quad , \quad u_L^j \mapsto e^{i\beta_j} u_L^j \\ d_R^j &\mapsto e^{i\alpha_j} d_R^j \quad , \quad u_R^j \mapsto e^{i\beta_j} u_R^j \end{aligned} \tag{34}$$

where there is *no* summation over j. This would have eliminated all the 6 phases, however if all the angles are equal $\alpha_j = \beta_j = \varphi$, then V does not change. Thus, only 5 phases can be eliminated this way. In the end we are left with 3 angles, and one phase.

The standard parametrization for the CKM matrix is

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{35}$$

where $c_{ij} := \cos(\varphi_{ij})$ and $s_{ij} = \sin(\varphi_{ij})$. The three angles $\{\varphi_{12}, \varphi_{23}, \varphi_{13}\}$ are rotation angles in the ij-flavour plane. The numerical values of the three angles and the phase are[6]

[6] R. L. Workman et al. (2022). In: *PTEP* 2022, p. 083C01.

$$\begin{aligned} \theta_{12} &= 0.22500 \pm 0.00067 = 12.892^\circ \pm 0.03839^\circ \\ \theta_{23} &= 0.01482^{+0.00085}_{-0.00074} = 2.39611^\circ{}^{+0.04870^\circ}_{-0.04240^\circ} \\ \theta_{13} &= 0.00369 \pm 0.00011 = 0.21142^\circ \pm 0.00630^\circ \\ \delta &= 1.144 \pm 0.027 = 65.55^\circ \pm 1.55^\circ . \end{aligned} \tag{36}$$

If the CKM matrix were real, there would be no CP violation. Therefore, the phase δ measures the amount of CP violation.

2 Strong CP violation

In the previous section, we have seen how to diagonalize the quark mass matrix, but we haven't talked about the consequences of this operation apart from the CKM matrix. Notice that the transformations that we have performed, in particular the ones in Eq. (31) are *chiral* meaning that right-handed and left-handed components transform independently. We know that these transformations are anomalous so as a result the Lagrangian is modified according to Eq. (2) due to the chiral anomaly. In this section our goal is to compute the modification of the QCD Lagrangian coming from the diagonalization of the quark mass matrix.

Chiral rotations with multiple generations

Let us start by deriving a result which we will be very useful shortly. Consider the right-handed $\{\psi_R^i\}$ and left-handed $\{\psi_L^i\}$ Weyl fermions where i is the generation index. Now consider the chiral rotations

$$\psi_R^i \mapsto R^{ij}\psi_R^j \quad \text{and} \quad \psi_L^i \mapsto L^{ij}\psi_L^j . \tag{37}$$

In matrix form

$$\boldsymbol{\psi}_R \mapsto R\,\boldsymbol{\psi}_R \quad \text{and} \quad \boldsymbol{\psi}_L \mapsto L\,\boldsymbol{\psi}_L . \tag{38}$$

Since L and R unitary, they can be diagonalized using unitary matrices. Then we can write

$$\boldsymbol{\psi}_R \mapsto W_R R_d W_R^\dagger\,\boldsymbol{\psi}_R \quad \text{and} \quad \boldsymbol{\psi}_L \mapsto W_L L_d W_L^\dagger\,\boldsymbol{\psi}_L , \tag{39}$$

where W_R and W_L are unitary, R_d and L_d are unitary and diagonal matrices. Now perform a change of basis under which $\vec{\boldsymbol{\psi}}_L \mapsto \vec{\boldsymbol{\psi}}'_L := W_L^\dagger \vec{\boldsymbol{\psi}}_L$ and similarly for $\vec{\boldsymbol{\psi}}_R$. In this basis, the transformations in Eq. (38) become

$$\vec{\boldsymbol{\psi}}'_R \mapsto R_d\,\vec{\boldsymbol{\psi}}'_R \quad \text{and} \quad \vec{\boldsymbol{\psi}}'_L \mapsto L_d\,\vec{\boldsymbol{\psi}}'_L . \tag{40}$$

We see that the transformations do not mix the flavors anymore. The moral of the story is we can always go to a base where a general chiral transformation parametrized by L and R becomes diagonal. So by introducing the Dirac fermions

$$\Psi_i = \begin{pmatrix} \psi_L^i \\ \psi_R^i \end{pmatrix}, \tag{41}$$

for each generation, we can write the transformation law for each generation as

$$\Psi_j \mapsto \begin{pmatrix} L_d^j & 0 \\ 0 & R_d^j \end{pmatrix} \Psi_i = \begin{pmatrix} e^{il_j} & 0 \\ 0 & e^{ir_j} \end{pmatrix} \Psi_j = \exp\left\{ i \begin{pmatrix} l_j & 0 \\ 0 & r_j \end{pmatrix} \right\} \Psi_j, \tag{42}$$

where L_d^j and R_d^j are the jth components of the L_d and R_d diagonal matrices, with l_j and r_j being real.[7] In order to bring this into the form $q \mapsto e^{i\alpha\gamma_5} q$ for which we know how the Lagrangian transforms, we can write the transformation in Eq. (42) as

[7] The components of L_d and R_d have unity amplitudes since they are unitary.

$$\Psi_j \mapsto \exp\left\{ i \frac{\alpha_j}{2} \gamma_5 \right\} \exp\left\{ i \frac{\beta_j}{2} \right\} \Psi_j, \tag{43}$$

with

$$\alpha_j = r_j - l_j \quad \text{and} \quad \beta_j = r_j + l_j. \tag{44}$$

The transformation with $e^{i\beta_j/2}$ is a vector rotation. Hence it is not anomalous, we don't need to deal with it.

Now from the anomaly result given in Eq. (2), we can deduce that the change in the Lagrangian from the full transformation is

$$\Delta\mathcal{L} = -\left(\sum_j \alpha_j \right) \times \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}. \tag{45}$$

The coefficient in front can be written as

$$\sum_j (r_j - l_j) = \arg \prod_j e^{ir_j} e^{-il_j} = \arg\det\left(L_d^\dagger R_d \right) = \arg\det\left(L^\dagger R \right). \tag{46}$$

The last equality follows from the fact the diagonalization in Eq. (39) is a similarity transformation, and similar matrices have the same determinant. So we have found that the rotation given in Eq. (38) modifies the Lagrangian by

$$\Delta\mathcal{L} = -\frac{g_s^2 \theta_q}{32\pi^2} G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}, \quad \theta_q := \arg\det\left(L^\dagger R \right). \tag{47}$$

Next we apply this result to quarks.

Phase induced by the diagonalization of the quark mass matrix

We have seen that the Yukawa matrices can be expressed as

$$Y_d = U_d M_d K_d^\dagger \quad \text{and} \quad Y_u = U_u M_u K_u^\dagger, \tag{48}$$

where M_d and M_u are diagonal and real, and the matrices $U_{u,d}$ and $K_{u,d}$ are unitary. Without loss of generality, we can also express them as

$$Y_d = U_d M_d U_d^\dagger \widetilde{K}_d^\dagger \quad \text{and} \quad Y_u = U_u M_u U_u^\dagger \widetilde{K}_u^\dagger, \tag{49}$$

where $\widetilde{K}_{u,d}$ are other unitary matrices. Then, the mass Lagrangian in Eq. (27) takes the form

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}\left(\mathbf{d}_L^\dagger\, U_d M_d U_d^\dagger \widetilde{K}_d^\dagger\, \mathbf{d}_R + \mathbf{u}_L^\dagger\, U_u M_u U_u^\dagger \widetilde{K}_u^\dagger\, \mathbf{u}_R\right) + \text{h.c.} \tag{50}$$

We can see that in order to go to the mass basis, we need to perform a chiral rotation first given by

$$\mathbf{d}_R \mapsto \widetilde{K}_d\, \mathbf{d}_R \quad \text{and} \quad \mathbf{u}_R \mapsto \widetilde{K}_u\, \mathbf{u}_R, \tag{51}$$

and then a non-chiral rotation with

$$\mathbf{d}_{L,R} \mapsto U_d\, \mathbf{d}_{L,R} \quad \text{and} \quad \mathbf{u}_{L,R} \mapsto U_u\, \mathbf{u}_{L,R}. \tag{52}$$

The non-chiral rotation is not anomalous so it doesn't modify the Lagrangian. The phase induced by the chiral rotations is

$$\theta_q = \arg\det \widetilde{K}_d \widetilde{K}_u = \arg\det\left(Y_d Y_u\right)^{-1} = -\arg\det Y_d Y_u, \tag{53}$$

where we have used Eq. (49) and the fact that the elements of M_d and M_u are real.

Physical θ parameter

By using our recently derived result, we can write our final expression for the θ term in the QCD:

$$\mathcal{L}_\theta = \frac{\bar{\theta} g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}, \quad \bar{\theta} := \theta_{\text{QCD}} - \theta_q = \theta_{\text{QCD}} + \arg\det Y_d Y_u. \tag{54}$$

Here θ_{QCD} is the θ parameter of QCD that is related to the θ vacua that we have studied in the last lecture, and θ_q comes from the diagonalization of the quark mass matrix. The significance of the $\bar{\theta}$ term is that it cannot be removed by a chiral transformation. To see this explicitly we note that in QCD the current associated to the chiral transformation is not conserved:

$$\partial_\mu J_5^\mu = 2 m_q \bar{q}\, i\gamma_5 q + \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu}. \tag{55}$$

The first term arises due to the non-zero quark masses, while the second term comes from the chiral anomaly. Under a chiral transformation $q \mapsto e^{i\alpha\gamma_5}q$ the Lagrangian is modified by, see Lecture 4

$$\begin{aligned}\Delta\mathcal{L} = -\alpha\partial_\mu J_5^\mu &= -2\alpha m_q \bar{q}\, i\gamma_5 q - \frac{2\alpha g_s^2}{32\pi^2} G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu} \\ &= -\bar{q} m_q e^{i2\alpha\gamma_5} q - \frac{2\alpha g_s^2}{32\pi^2} G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}.\end{aligned} \tag{56}$$

Now we can perform a chiral rotation with $\alpha = \bar{\theta}/2$ and this will remove the $\bar{\theta}G\widetilde{G}$ term from the Lagrangian, but due to the first term in Eq. (56) it will appear in the quark mass term

$$-\bar{q}m_q q \mapsto \bar{q}m_q\left(1 + e^{i\bar{\theta}\gamma_5}\right)q. \tag{57}$$

For this reason $\bar{\theta}$ becomes a physical parameter and will have an effect on the physical phenomena as we shall see soon.

3 Strong CP Problem

We are now ready to state the Strong CP problem. The electric dipole moment of the neutron is defined by the Hamiltonian

$$H = -d_n \mathbf{E}\cdot\hat{\mathbf{S}}. \tag{58}$$

A non-zero dipole moment has not been measured by any expriment yet. The current experimental limit is[8]

$$\left|d_n^{\text{exp}}\right| < 1.8\times 10^{-26}\ e\ \text{cm} \quad (90\%\ \text{CL}). \tag{59}$$

The most precise theoretical calculation for d_n based on the QCD sum rules is[9]

$$d_n = 2.4(1.0)\times 10^{-16}\ \bar{\theta}\ e\ \text{cm} = 1.2(0.5)\times 10^{-2}\ \bar{\theta}\ e\ \text{GeV}^{-1}. \tag{60}$$

We see that the neutron electric dipole moment is proportional to the $\bar{\theta}$ parameter, hence it should be physical. Comparing Eqs. (59) and (60) yields the bound

$$\left|\bar{\theta}\right| \lesssim 10^{-10}. \tag{61}$$

Why the $\bar{\theta}$ parameter is so small if not zero is called the Strong CP Problem.

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