Lecture 5: Instantons and Theta vacua

Cem Eröncel

cem.eroncel@.itu.edu.tr

Lecture date: March, 21 2023

Last update: April 5, 2023

In the last lecture, we have seen that a chiral rotation $q \mapsto e^{i\alpha\gamma_5}q$ with a single quark q modifies the QCD Lagrangian by

$$\mathcal{L}_{\text{QCD}} \mapsto \mathcal{L}_{\text{QCD}} - \frac{\alpha g_s^2}{16\pi^2} G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}. \tag{1}$$

But, we have also mentioned that this term can be written as a total derivative of the so called Chern-Simons current $\mathcal{K}^\mu$:

$$G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu} = \partial_\mu \mathcal{K}^\mu = \partial_\mu \epsilon^{\mu\alpha\beta\gamma}\left(A^a_\alpha G^a_{\beta\psi} - \frac{g_s}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\gamma\right). \tag{2}$$

Therefore, we might naively think that this term is irrelevant, since it does not contribute to the equations of motion. However, this statement is true only at the *perturbative* level. This term crucially affects the vacuum structure of QCD, albeit *non-perturbatively*. Furthermore, this effect does not depend on whether the gauge coupling constant g_s is small or not. Non-perturbative effects can be present even if the theory is weakly coupled.

You might be suprised about the last statement but probably you have already encountered such an example in one of your Quantum Mechanics courses. This is the phenomenon of transmission through a potential barrier. Using the WKB approximation, the amplitude for the transmission is found to be

$$|\mathcal{T}(E)| = \exp\left\{-\frac{1}{\hbar}\int_{x_1}^{x^2} \mathrm{d}x\,\sqrt{2(V-E)}\right\}[1+\mathcal{O}(\hbar)], \tag{3}$$

where $E < V_{\max}$ is the energy of the particle with x_1 and x_2 being the corresponding *classical turning points*. No matter how small the coupling in the potential V is, this effect can not be seen in any order of perturbation theory. Hence, it is strictly a non-perturbative phenomenon.

They are phenomenon also in Quantum Field Theory that are analogs of barrier penetration problem in Quantum Mechanics. This is the topic of this and the next week's lectures. At the end, we will understand why the term in Eq. (2) cannot simply be ignored.

In this lecture we will be covering instantons in quantum mechanics. Next week, we will discuss instantons in Quantum Field Theory, and also discover the non-trivial vacuum structure of the QCD. For more details about these topics, we refer the reader to the lectures by Sidney Coleman,[1] Hilmar Forkel,[2] and David Tong.[3]

[1] Sidney Coleman (1985). "The uses of instantons". In: *Selected Erice lectures. Aspects of Symmetry*. Cambridge University Press.

[2] Hilmar Forkel (Aug. 2000). In: arXiv: hep-ph/0009136.

[3] David Tong (2018). *Gauge Theory*.

1 Instantons in Quantum Mechanics

Before dealing with the vacuum structure of the QCD, let us tackle a simpler problem in Quantum Mechanics where we can get a feel of instantons.

We consider a particle of unit mass in one dimension with the Hamiltonian

$$H = \frac{p^2}{2} + V(x). \tag{4}$$

By performing a Wick rotation, we can write the imaginary time $\tau := it$ version of the Feynman's sum over histories:

$$\left\langle x_f \middle| e^{-HT} \middle| x_i \right\rangle = \mathcal{N} \int [\mathrm{d}x] e^{-S_E/\hbar}. \tag{5}$$

On the left side, the states $|x_i\rangle$ and $\left|x_f\right\rangle$ are the initial and final states at $\tau = -T/2$ and $\tau = T/2$ respectively. By inserting a complete set of energy eigenstates $\{|n\rangle\}$ obeying $H|n\rangle = E_n|n\rangle$ it can be written as

$$\left\langle x_f \middle| e^{-HT} \middle| x_i \right\rangle = \sum_n e^{-E_n T/\hbar} \left\langle x_f \middle| n \right\rangle \langle n | x_i \rangle . \tag{6}$$

The leading term in this expansion at large T gives the eigenvalue and the wavefunction of the lowest energy eigenstate.

On the right side of Eq. (5), there is a sum over all paths $x(\tau)$ that obey the boundary conditions

$$x(-T/2) = x_i \quad \text{and} \quad x(T/2) = x_f. \tag{7}$$

The factor $\mathcal{N}$ is a normalization factor, and the S_E is the Euclidean action:

$$S_E = \int_{-T/2}^{T/2} \mathrm{d}\tau \left[\frac{1}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 + V \right]. \tag{8}$$

We see that this has the same form as the Minkowski action in physical time, but the sign of the potential is flipped. This implies that the minima of the potential, i.e. the classical ground states in physical time, correspond to the maxima of the inverted potential.

In the semi-classical limit where $\hbar \to 0$, the path integral on the RHS in Eq. (5) is dominated by the stationary points of the Euclidean action. The corresponding equation of motion reads

$$x''(\tau) - V'(x) = 0. \tag{9}$$

Not suprisingly, this is equivalent to the classical equation of motion of the particle in the inverted potential $-V$. In the following, we will see what this implies.

2 Double-well potential

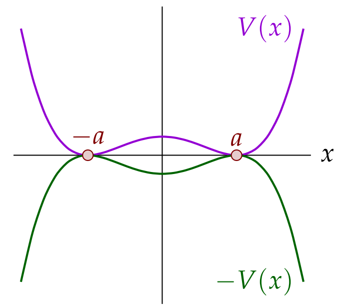

Figure 1: The double-well potential, and its inverted version.

Consider a potential that has two minima located at $x = \pm a$ as shown in Figure 1. These correspond to the ground states of the system. In Euclidean time, the potential is inverted so the minima become hill tops. Suppose a particle is initially in one of the ground states, or hill tops in Euclidean time. What are the possible solutions to the Euclidean equation of motion?

1. The particle can stay at the hill top where it started.

2. It can travel from one hill top to the other.

In physical time, the first solution corresponds to the situation where a particle initially at one of the ground states and *stays* at the same ground state. This is what we expect classically. The second one is much more interesting. It corresponds to the situation where the particle *tunnels* from one minimum to the other. So the phenomenon of quantum tunneling which is not a classical solution in physical time turns out to be a classical solution in Euclidean time!

Consider a scenario where the particle starts, say, at the left hill $x = -a$ at $T = -\infty$, and reaches to the right hill $x = a$ at $T = +\infty$. The Euclidean "energy" of the particle

$$E = \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - V(x) \tag{10}$$

is conserved and equal to zero by assuming that $V = 0$ at the minima. Thus, we can write

$$\tau = \tau_1 + \int_0^x \mathrm{d}x' \left[2V(x')\right]^{-1/2}, \tag{11}$$

where τ_1 is an integration constant that denotes the Euclidean time at which x vanishes. This solution is known as an instanton with the center at τ_1. Similarly, we can construct solutions that start at the right hill $x = a$ at $T = -\infty$, and arrive to the left hill $x = -a$ at $T = +\infty$. These are called anti-instantons. A sketch of the solutions is shown in Figure 2.

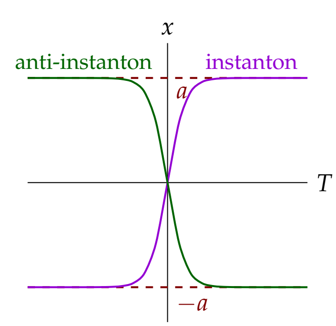

Figure 2: An example of the instanton and anti-instanton solution.

References

Coleman, Sidney (1985). "The uses of instantons". In: *Selected Erice lectures*. *Aspects of Symmetry*. Cambridge University Press.

Forkel, Hilmar (Aug. 2000). "A Primer on instantons in QCD". In: arXiv: hep-ph/0009136.

Tong, David (2018). *Gauge Theory*. URL: https://www.damtp.cam.ac.uk/user/tong/gaugetheory/gt.pdf.