Lecture 4: Anomalies

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In the last lectures we have discussed the symmetries of the QCD with massless quarks:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_q i\,\bar{q} \not{D} q. \tag{1}$$

We have seen that this model enjoys various symmetries. With N being the number of massless quarks, these are

- **Vector rotations** individually rotate the phases of the quarks:

$$q \mapsto e^{i\alpha} q, \quad \alpha \in \mathbb{R}. \tag{2}$$

- **Chiral rotations** act on the N-vectors constructed by the left- and right-handed components of the quarks:

$$\mathbf{q}_L \mapsto L\,\mathbf{q}_L, \quad \mathbf{q}_R \mapsto R\,\mathbf{q}_R, \quad L \in \mathsf{SU}(N)_L \quad \text{and} \quad R \in \mathsf{SU}(N)_R. \tag{3}$$

- **Axial rotations** are like vector rotations but they rotate the left- and right-handed components with opposite phases:[1]

$$q \mapsto e^{i\alpha\gamma_5} q. \tag{5}$$

[1] It is useful to remember that $\gamma^5 \equiv \gamma_5$ can be defined in a basis-independent way via

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

We have also briefly mentioned that the last of these symmetries is not a real symmetry of the Lagrangian, since it is broken by the quantum effects. This phenomenon is called an anomaly. An anomalous symmetry is a symmetry of the classical theory which does not survive the quantum theory. In this lecture, we will see how this occurs, and also show that the anomaly requires us to add another term to the QCD Lagrangian given in Eq. (1).

We will closely follow the discussion in the lecture notes by David Tong.[2] Other resources that cover more formal concepts in anomalies are the lectures by Adel Bila,[3] and the textbook by Bertlmann.[4]

[2] David Tong (2018). *Gauge Theory*.

[3] Adel Bilal (Feb. 2008). In: arXiv: 0802.0634 [hep-th].

[4] Reinhold A. Bertlmann (2001). *Anomalies in Quantum Field Theory*. Revised ed. edition. Oxford: Oxford University Press. 584 pp.

1 Noether theorem

We start by recalling the Noether theorem which relates the symmetries of the Lagrangians to conserved currents. Let $\mathcal{L} = \mathcal{L}[\psi_n, \partial_\mu \psi_n]$ denotes a Lagrangian for n fields $\{\psi_n\}$ that are not necessarily scalars.

We assume that this Lagrangian is invariant under a continuous global symmetry parametrized by ϵ. Since the symmetry is continuous, we can consider an infinitesmall transformation and write

$$\psi_n(x) \mapsto \psi'_n(x) = \psi_n(x) + \epsilon X_n(\psi). \tag{6}$$

The Noether theorem can be proven in different ways. We will follow a method which will also be useful when deriving the Ward identities in the next section. For this, we promote the constant ϵ to a continuous parameter $\epsilon(x)$ and write

$$\psi_n(x) \mapsto \psi'_n(x) = \psi_n(x) + \epsilon(x) X_n(\psi). \tag{7}$$

Under this transformation, the Lagrangian is not necessarily invariant, but it transforms as $\mathcal{L} \to \delta\mathcal{L}$ where

$$\begin{aligned}\delta\mathcal{L} &= \sum_n \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \delta(\partial_\mu \psi_n) + \frac{\partial \mathcal{L}}{\partial \psi_n} \delta\psi_n \right\} \\ &= \sum_n \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \partial_\mu(\epsilon(x) X_n(\psi)) + \frac{\partial \mathcal{L}}{\partial \psi_n} \epsilon(x) X_n(\psi) \right\}, \\ &= \sum_n \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} (\partial_\mu \epsilon) X_n(\psi) + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \partial_\mu X_n(\psi) + \frac{\partial \mathcal{L}}{\partial \psi_n} X_n(\psi) \right] \epsilon \right\}.\end{aligned} \tag{8}$$

Now if $\mathcal{G}$ is a symmetry of the theory, then it can modify the Lagrangian only up to a total derivative, i.e.

$$\delta\mathcal{L} = \epsilon \partial_\mu \mathcal{F}^\mu. \tag{9}$$

This should be identical to Eq. (8) when ϵ is constant which means we can replace the term inside the square bracket in Eq. (8) to arrive at

$$\delta\mathcal{L} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} (\partial_\mu \epsilon) X_n(\psi) + \epsilon \partial_\mu \mathcal{F}^\mu. \tag{10}$$

Then the corresponding change in the action reads[5]

$$\begin{aligned}\delta S = \int \mathrm{d}^4x\, \delta\mathcal{L} &= \int \mathrm{d}^4x \left\{ \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} (\partial_\mu \epsilon) X_n(\psi) + \epsilon \partial_\mu \mathcal{F}^\mu \right\} \\ &= -\int \mathrm{d}^4x\, \epsilon(x) \partial_\mu \left[\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} X_n(\psi) - \mathcal{F}^\mu \right].\end{aligned} \tag{11}$$

[5] When arriving to the second step in Eq. (11) we assumed that $\epsilon(x)$ vanishes sufficiently fast at infinity so that there is no contribution from the boundary.

If $\{\psi_n\}$ obey the classical equations of motion then $\delta S = 0$ for *any* change $\delta\psi_n$ including the transformation (7). This implies that when the classical equations of motion are satisfied, we have the conservation law

$$\partial_\mu J^\mu = 0, \quad \text{where} \quad J^\mu = \sum_n \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} X_n \right) - \mathcal{F}^\mu. \tag{12}$$

This is the Noether theorem, and J^μ is called the Noether current. The total charge Q given by

$$Q = \int d^3x\, J^0 \tag{13}$$

is called the Noether charge, and is conserved because

$$\dot{Q} := \partial_0 Q = \int d^3x\, \partial_0 J^0 = -\int d^3x\, \nabla \cdot \mathbf{J} = -\oint (\mathbf{J} \cdot \hat{\mathbf{n}})\, d\mathbf{s} = 0, \tag{14}$$

assuming that the current vanishes at infinity.

This was all classical results. Let us see which of these statements remain to be valid in the quantum theory.

2 Ward Identities in Quantum Field Theory

In this section, we will derive an analog of the Noether theorem in Quantum Field Theory. We will consider a theory with a single scalar field ϕ for simplicity, but the main arguments are identical for more complex theories.

In Quantum Field Theory, all the information about a system is encoded in the so called generating functional given by the path integral

$$Z[K] = \int \mathcal{D}\phi \exp\left\{ iS[\phi] + i \int d^4x\, K(x)\phi(x) \right\}, \tag{15}$$

where $K(x)$ is an external classical source for $\phi(x)$[6], and $\mathcal{D}\phi$ is an integral over all field configurations, not just the ones that satisfy the equations of motion. We will assume that there is a continuous symmetry transformation $\mathcal{G}$ whose infinitesmall form acts on ϕ as

[6] Standard notation for the current is J. Here we are using K in order to reserve J for the Noether current.

$$\mathcal{G} : \phi \mapsto \phi' = \phi + \epsilon X(\phi), \tag{16}$$

where ϵ is an infinitesmall parameter. Now we transform ϵ to a continuous parameter $\epsilon(x)$ and write

$$\mathcal{G} : \phi \mapsto \phi' = \phi + \epsilon(x) X(\phi). \tag{17}$$

Since this is just a field redefinition, the generating functional (15) can be written in terms of ϕ' as well:

$$Z[K] = \int \mathcal{D}\phi' \exp\left\{ iS[\phi'] + i \int d^4x\, K(x)\phi'(x) \right\}. \tag{18}$$

From Eq. (11) we can deduce that the modified action $S[\phi']$ is related to the original action $S[\phi]$ via

$$S[\phi'] = S[\phi] - \int d^4x\, \epsilon(x) \partial_\mu J^\mu(x), \tag{19}$$

where $J^\mu(x)$ is the Noether current corresponding to the symmetry $\mathcal{G}$. Then, Eq. (18) becomes

$$\begin{aligned} Z[K] = \int \mathcal{D}\phi' \exp\left\{ iS[\phi] + i \int \mathrm{d}^4x\, K(x)\phi(x) \right\} \\ \times \exp\left\{ -i \int \mathrm{d}^4x\, \epsilon(x) \big(\partial_\mu J^\mu(x) - K(x)X(\phi) \big) \right\}. \end{aligned} \tag{20}$$

Since $\epsilon(x)$ is infinitesmall, we can expand the second expontential and obtain

$$\begin{aligned} Z[K] = \int \mathcal{D}\phi' \exp\left\{ iS[\phi] + i \int \mathrm{d}^4x\, K(x)\phi(x) \right\} \\ \times \left[1 - i \int \mathrm{d}^4x\, \epsilon(x) \big(\partial_\mu J^\mu(x) - K(x)X(\phi) \big) \right]. \end{aligned} \tag{21}$$

To proceed further we will make the *assumption* that the integral measure $\mathcal{D}\phi$ does not change under the symmetry transformation:

$$\textbf{Assumption:} \quad \mathcal{D}\phi = \mathcal{D}\phi'. \tag{22}$$

With this assumption, the first term in Eq. (21) is identically the generating functional $Z[K]$. Thus

$$0 = \int \mathcal{D}\phi \exp\left\{ iS[\phi] + i \int \mathrm{d}^4x\, K\phi \right\} \int \mathrm{d}^4x\, \epsilon(x) \big(\partial_\mu J^\mu - K\,X(\phi) \big). \tag{23}$$

This expression should be true for all $\epsilon(x)$ so we can get rid of the space integral and write

$$0 = \int \mathcal{D}\phi \exp\left\{ iS[\phi] + i \int \mathrm{d}^4x\, K\phi \right\} \big(\partial_\mu J^\mu - K\,X(\phi) \big) \tag{24}$$

By setting $K = 0$ we find

$$\int \mathcal{D}\phi \exp\{iS[\phi]\} \big(\partial_\mu J^\mu \big) = \langle \partial_\mu J^\mu \rangle = 0. \tag{25}$$

Differentiating Eq. (24) multiple times with respect to K and then setting $K = 0$ gives the following formula:

$$\partial_\mu \left\langle J^\mu(x)\phi(x^1)\phi(x^2)\cdots\phi(x^n) \right\rangle = 0 \quad \text{if} \quad x \neq x^i. \tag{26}$$

These are known as the Ward identities and they mean that $\partial_\mu J^\mu$ vanishes inside any correlation function as long as its position does not coincide with the insertion point of other fields. This is the quantum analog of the Noether theorem. If this identity is satisfied, then it means that the classical symmetry surives in the quantum theory.

When deriving this result, our only non-trivial assumption was the invariance of the measure, i.e. Eq. (22). Therefore, we expect that the presence of an anomaly is related to the invariance of the path integral measure. This is indeed the case, and in the next sections we will see explicit examples of these phenomenon.

3 The Chiral Anomaly

We start with a simpler example to discuss the anomaly. We consider a massless Dirac fermion ψ coupled to electromagnetism. The action is

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) + \int \mathrm{d}^4 x \, i \, \overline{\psi} \not{D} \psi, \tag{27}$$

where $\not{D}$ is the gauge-covariant derivative contracted with the gamma matrices:

$$\not{D} := \gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu - ieQA_\mu\right), \tag{28}$$

where Q is the EM charge of the fermion normalized such that $Q = -1$ for the electron.[7] This action has two *global* symmetries:

- **Vector rotations** $V : \psi \mapsto e^{i\epsilon}\psi$ with the corresponding Noether current
$$J^\mu_V = \overline{\psi} \gamma^\mu \psi. \tag{29}$$
- **Axial rotations** $A : \psi \mapsto e^{i\epsilon\gamma_5}\psi$ with the Noether current
$$J^\mu_A = \overline{\psi} \gamma^\mu \gamma^5 \psi. \tag{30}$$

[7] In this notation, the fermion ψ transforms under the $\mathrm{U}(1)_{\mathrm{EM}}$ gauge transformation as
$$\psi \mapsto \exp\{iQ\alpha(x)\}\psi.$$

Classically, the Noether theorem tells us

$$\partial_\mu J^\mu_V = 0 \quad \text{and} \quad \partial_\mu J^\mu_A = 0. \tag{31}$$

We want to learn which of these classical statements survive in the quantum theory.

In the previous section we saw that whether a symmetry is still applicable in the quantum theory is related to the invariance of the path integral measure under the symmetry transformation. In this case we are interested in the transformation of the fermion measure

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi. \tag{32}$$

Under an infinitesmall vector and axial transformations, ψ and $\overline{\psi}$ change by

$$V : \psi \mapsto \psi' = \psi + i\epsilon\psi, \quad V : \overline{\psi} \mapsto \overline{\psi}' = \overline{\psi} - i\epsilon\overline{\psi}, \tag{33}$$
$$A : \psi \mapsto \psi' = \psi + i\epsilon\gamma^5\psi, \quad A : \overline{\psi} \mapsto \overline{\psi}' = \overline{\psi} + i\epsilon\overline{\psi}\gamma^5, \tag{34}$$

The crucial observation is that with the vector symmetry the transformations of ψ and $\overline{\psi}$ differ by a sign, while this is not the case for the axial symmetry A. This will be the determining factor behind the anomaly of the axial symmetry.

Euclidean path integral

For the anomaly calculations, it is more convenient to perform a Wick rotation and switch to the Euclidean coordinates via introducing

$$x^4 := -ix^0 = -it. \tag{35}$$

After this the metric becomes

$$g_{\mu\nu} \to -\delta_{\mu\nu}\, \mathrm{d}x_E^\mu\, \mathrm{d}x_E^\nu\,, \tag{36}$$

where the subscript E denotes the Euclidean coordinates. This transformation requires the introduction of the corresponding components of the gamma matrix, partial derivative, and the vector potential via

$$\gamma^4 := i\gamma^0, \quad \partial_4 := -i\partial_0, \quad A_4 := -iA_0. \tag{37}$$

In order to distinguish between the gamma matrices in Minkowski space we introduce the notation

$$\gamma^\mu = \left\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\right\}, \quad \gamma_E^\mu = \left\{\gamma^1, \gamma^2, \gamma^3, \gamma^4\right\}. \tag{38}$$

In Euclidean space, all the gamma matrices are anti-hermitian, i.e.

$$\left(\gamma_E^\mu\right)^\dagger = -\gamma_E^\mu. \tag{39}$$

However, γ^5 remains Hermitian:

$$\left(\gamma_E^5\right)^\dagger = \gamma_E^5, \tag{40}$$

where

$$\gamma^5 = \gamma_E^5 = \gamma^4\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4. \tag{41}$$

The last equality comes from anti-commuting the gamma matrices.

Under a Wick rotation the weight of the path integral e^{iS} is transformed to e^{-S_E} where S_E is the Euclidean action. The latter is defined by performing the Wick rotation in S and then setting $S_E := -iS$.

Defining the fermion measure

We want to know how the measure given in Eq. (32) is modified under the transformations given in Eqs. (33) and (34). But before doing this, we need to properly define the measure. For this we consider the Dirac operator $\not{D}$ for a Dirac spinor in the background of an electromagnetic field $A_\mu(x)$. This operator satisfies an eigenvalue equation:

$$\not{D}\psi_n = \lambda_n \psi_n, \tag{42}$$

where $\{\lambda_n\}$ are the eigenvalues and $\{\psi_n\}$ are the eigenspinors. Note that in Euclidean space, the Dirac operator

$$\not{D}_E := \gamma_E^\mu D_\mu, \quad \mu = 1, 2, 3, 4, \tag{43}$$

is hermitian $\not{D}_E^\dagger$. This implies that the eigenvalues $\{\lambda_n\}$ are real, and the eigenspinors are orthonormal is the sense that

$$\int \mathrm{d}^4x_E\, \psi_n^\dagger(x)\psi_m(x) = \delta_{nm}, \tag{44}$$

and they satifsy the completeness relation

$$\sum_n \psi_n(y)\psi_n^\dagger(x) = \sum_n \langle y|n\rangle \langle n|x\rangle = \langle y|x\rangle = \delta^{(4)}(y-x). \tag{45}$$

A general Dirac spinor ψ can be expressed in terms of eigenspinors as

$$\psi(x) = \sum_n \alpha_n\psi_n(x) = \sum_n a_n \langle x|n\rangle \tag{46}$$

$$\overline{\psi}(x) = \sum_n \psi_n^\dagger(x)\overline{\beta}_n = \sum_n \langle n|x\rangle\, \overline{\beta}_n. \tag{47}$$

where $\{\alpha_n\}$ and $\{\overline{\beta}_n\}$ are Grassmann-valued numbers[8] so that the spinors can satify the anti-commutation relations. Thus, we can write the Euclidean action for the Dirac fermions as

[8] Grassmann numbers are reviewed in Appendix A.

$$S_E = -i\int \mathrm{d}^4x_E\, \overline{\psi}(x)\not{D}_E\psi(x) = -i\sum_n \lambda_n\overline{\beta}_n\alpha_n. \tag{48}$$

Then, we can define the fermion measure as

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \longrightarrow \prod_n \int \mathrm{d}\overline{\beta}_n\, \mathrm{d}\alpha_n\,. \tag{49}$$

So the Euclidean partition function for the fermions reads

$$Z_f^E = \prod_n \int \mathrm{d}\overline{\beta}_n\, \mathrm{d}\alpha_n \exp\left\{ i\sum_m \lambda_m\overline{\beta}_m\alpha_m \right\}. \tag{50}$$

We now remember that the Grassmann integrations are very easy. They are given by

$$\int \mathrm{d}\alpha = 0, \quad \int \mathrm{d}\alpha\, \alpha = 1, \tag{51}$$

and similarly for $\overline{\beta}$. Furthermore the square of any Grassmann number vanishes due to the anti-commutativity. So, we can directly evaluate the Euclidean partition function as[9]

[9] At the second step of Eq. (52) the extra minus sign comes from anti-commuting $\mathrm{d}\alpha_m$ and $\overline{\beta}_m$.

$$\begin{aligned} Z_f^E &= \prod_n \int \mathrm{d}\overline{\beta}_n\, \mathrm{d}\alpha_n \left(1 + i\sum_m \lambda_m\overline{\beta}_m\alpha_m\right) \\ &= i\prod_n\sum_m \lambda_m \left(\int \mathrm{d}\overline{\beta}_n\, \overline{\beta}_m\right)\left(\int \mathrm{d}\alpha_n\, \overline{\alpha_m}\right) \\ &= \sum_n \lambda_n =: \det\{i\not{D}_E\}. \end{aligned} \tag{52}$$

Note that this expression is *exact*. When expanding the exponential in Eq. (50) only first two terms survive.

Calculating the Jacobian

Let us return to the calculation of the transformation of the fermion measure under the vector and axial rotations. We start with the axial rotations since they will give interesting results. It is sufficient to consider the transformation of ψ since the transformation of $\overline{\psi}$ is identical for axial rotations, and differs by a sign for vector rotations.

Under an axial rotation

$$\delta\psi(x) = i\epsilon(x)\gamma^5\psi(x) \;\Rightarrow\; \sum_n \delta\alpha_n\psi_n(x) = i\epsilon(x)\sum_m \alpha_m\gamma^5\psi_m(x). \tag{53}$$

By applying the orthogonality relation (44) on both sides yields

$$\delta\alpha_n = i\int \mathrm{d}^4x_E\,\epsilon(x)\psi_n^\dagger(x)\sum_m \gamma^5\psi_m(x)\alpha_m =: X_{nm}\alpha_m. \tag{54}$$

We see that the transformation is linear. In matrix form

$$\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}' = (1+X)\boldsymbol{\alpha}, \tag{55}$$

where $\alpha_m := (\boldsymbol{\alpha})_m$. If we were dealing with the regular c-numbers, then the Jacobian $\mathcal{J}$ of this transformation would be

$$\mathcal{J} = \det(1+X), \quad c\text{-numbers} \quad . \tag{56}$$

For Grassmann variables, we instead have[10]

[10] See Appendix A for a derivation.

$$\mathcal{J} = \det{}^{-1}(1+X). \tag{57}$$

For $\overline{\psi}$, the transformation is the same so

$$\prod_n \int \mathrm{d}\overline{\beta}_n\,\mathrm{d}\alpha_n = \prod_n \int \mathrm{d}\overline{\beta}'_n\,\mathrm{d}\alpha'_n\,\mathcal{J}^2. \tag{58}$$

At leading order in ϵ, we can approximate the Jacobian as

$$\mathcal{J} = \det{}^{-1}(1+X) \simeq \det(1-X) \simeq \det\left(e^{-X}\right) = e^{-\operatorname{Tr}X}. \tag{59}$$

Explicitly,[11]

[11] Note that the Tr in Eq. (59) gets replaced by the sum in Eq. (60) since $\operatorname{Tr}X = \sum_n X_{nn}$.

$$\mathcal{J} = \exp\left\{-i\int \mathrm{d}^4x_E\,\epsilon(x)\sum_n \psi_n^\dagger(x)\gamma^5\psi_n(x)\right\}. \tag{60}$$

What remains left is to calculate this Jacobian.

Before proceeding with the evaluation of Eq. (60) let us briefly mention what will be different for vector transformations. Following similar steps it is straightforward to show that for the vector transformation, the Jacobian factors for ψ and $\overline{\psi}$ reads $\det^{-1}(1+Y)$ and $\det^{-1}(1-Y)$ respectively, where Y is same as X except that the γ^5

term is absent. The opposite sign is a direct consequence of the opposite sign in the infinitesmall transformations given in Eq. (33). Therefore, the combined Jacobian factor for the vector transformation

$$\mathcal{J}_V = \det{}^{-1}(1+Y)\det{}^{-1}(1-Y) \simeq \det(1+Y)\det(1-Y) \simeq 1 + \mathcal{O}(\epsilon^2) \tag{61}$$

is equal to the identity matrix at leading order in ϵ. This is sufficient in order not to contribute to the Ward identities, thus there is no anomaly with respect to the vector symmetry.

Let us return to the evaluation of the Jacobian given in Eq. (60). They are two naive guesses which we can make:

1. The infinite sum in Eq. (60) is proportional to the trace of γ^5 which is zero. Thus the Jacobian should vanish: $\mathcal{J} = 0$. [12]
2. In Eq. (60) we are summing over an infinite amount of modes at each point in space. There is no hope that the sum will converge, so $\mathcal{J} = \infty$.

[12] To see this take $\sum_n \overline{\psi}_n(x)\gamma^5\psi_n(x)$, integrate over d^4x, and use the completeness relation (44).

Of course, in reality both of these will play against each other, and in the end we will obtain a finite result. To see this, our first step is to regularize the infinite sum. We need to do this in a gauge invariant way. Since the eigenvalues $\{\lambda_n\}$ of the Dirac operator are gauge invariant, we can use them for regularization. So we write

$$\begin{aligned}\int \mathrm{d}^4x_E\,\epsilon(x)\sum_n \psi_n^\dagger\gamma^5\psi_n &= \lim_{\Lambda\to\infty}\int \mathrm{d}^4x_E\,\epsilon(x)\sum_n \psi_n^\dagger\gamma^5\psi_n e^{-\lambda_n^2/\Lambda^2} \\ &= \lim_{\Lambda\to\infty}\int \mathrm{d}^4x_E\,\epsilon(x)\sum_n \psi_n^\dagger\gamma^5 e^{-\not{D}_E^2/\Lambda^2}\psi_n \\ &=: \lim_{\Lambda\to\infty}\int \mathrm{d}^4x_E\,\epsilon(x)\mathcal{W}_\Lambda\end{aligned} \tag{62}$$

Now we take the Fourier transform of ψ_n via[13]

[13] Note that the momentum space is also Euclidean.

$$\psi_n(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{ik\cdot x}\psi(k), \tag{63}$$

so that $\mathcal{W}_\Lambda$ becomes

$$\mathcal{W}_\Lambda = \sum_n\int \frac{\mathrm{d}^4k\,\mathrm{d}^4k'}{(2\pi)^8}\psi_n^\dagger(k)e^{-ik\cdot x}\gamma^5 e^{-\not{D}_E^2/\Lambda^2}e^{ik'\cdot x}\psi_n(k'). \tag{64}$$

Again, by integrating this over d^4x_E we can see that

$$\mathcal{W}_\Lambda = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\Big(e^{-ik\cdot x}\gamma^5 e^{-\not{D}_E^2/\Lambda^2}e^{ik\cdot x}\Big). \tag{65}$$

They are two useful identities that we will need in the rest of the calculation. The first one is[14]

[14] From now on we take $Q = 1$ in order to not to clutter the notation. We can always take $e \to eQ$ to generalize the results derived in this section.

$$\begin{aligned} \not{D}_E^2 &= \gamma_E^\mu \gamma_E^\nu D_\mu D_\nu \\ &= \frac{1}{2}\left\{\gamma_E^\mu, \gamma_E^\nu\right\} D_\mu D_\nu + \frac{1}{2}\left[\gamma_E^\mu, \gamma_E^\nu\right] D_\mu D_\nu \\ &= D^\mu D_\mu + \frac{1}{4}\left[\gamma_E^\mu, \gamma_E^\nu\right]\left[D_\mu, D_\nu\right] \\ &= D^\mu D_\mu - \frac{ie}{2}\gamma_E^\mu \gamma_E^\nu F_{\mu\nu}, \end{aligned} \tag{66}$$

while the second one reads

$$e^{-ik\cdot x} D_\mu e^{ik\cdot x} = D_\mu + ik_\mu. \tag{67}$$

By combining these we get[15]

$$\begin{aligned} e^{-ik\cdot x} e^{-\not{D}_E^2/\Lambda^2} e^{ik\cdot x} &= e^{-ik\cdot x} \exp\left\{-\frac{1}{\Lambda^2}\left[D^\mu D_\mu - \frac{ie}{2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right]\right\} e^{ik\cdot x} \\ &= \exp\left\{-\frac{1}{\Lambda^2}\left[\left(D_\mu + ik_\mu\right)^2 - \frac{ie}{2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right]\right\}. \end{aligned} \tag{68}$$

From the Eq. (62) we have introduced the regulator Λ, we can see that its has mass dimension one. Since we will integrate over the momentum variables in Eq. (65), it will make sense to introduce the dimensionless momentum $\widetilde{k} := k/\Lambda$ with $k^2 = -k_\mu k^\mu$ [16] and write[17]

$$\begin{aligned} e^{-ik\cdot x} e^{-\not{D}_E^2/\Lambda^2} e^{ik\cdot x} &= \exp\left\{-\left(\frac{D_\mu}{\Lambda} + i\widetilde{k}_\mu\right)^2 + \frac{ie}{2\Lambda^2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right\} \\ &= \exp\left\{-\widetilde{k}^2 - \frac{2i\widetilde{k}^\mu D_\mu}{\Lambda} - \frac{D^2}{\Lambda^2} + \frac{ie}{2\Lambda^2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right\} \\ &= e^{-\widetilde{k}^2}\exp\left\{-\frac{2i\widetilde{k}^\mu D_\mu}{\Lambda} - \frac{D^2}{\Lambda^2} + \frac{ie}{2\Lambda^2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right\}. \end{aligned} \tag{69}$$

Now, Eq. (65) takes the form

$$\mathcal{W}_\Lambda = \Lambda^4 \int \frac{\mathrm{d}^4\widetilde{k}}{(2\pi)^4} e^{-\widetilde{k}^2}\,\mathrm{Tr}\left(\gamma^5 \exp\left\{-\frac{2i\widetilde{k}^\mu D_\mu}{\Lambda} - \frac{D^2}{\Lambda^2} + \frac{ie}{2\Lambda^2}\gamma_E^\mu\gamma_E^\nu F_{\mu\nu}\right\}\right) \tag{70}$$

Expanding the exponentials will give a quite complicated expression, however we need to keep only the terms that will survive once we take the $\Lambda \to \infty$ limit. Since introducing dimensionless momenta did bring a factor of Λ^4 into the measure of the momentum integral, we only need to keep the terms up to order Λ^{-4}.

We will also take the trace, so we also need to be aware of the gamma matrix structure. Most of the products involving gamma matrices with γ^5 have vanishing trace except

$$\mathrm{Tr}\left(\gamma^5\gamma_E^\mu\gamma_E^\nu\gamma_E^\rho\gamma_E^\sigma\right) = -4\epsilon^{\mu\nu\rho\sigma}, \tag{71}$$

with the convention $\epsilon^{1234} = \epsilon^{1230} = +1$. Thus, only terms from Eq. (70) that will survive the $\Lambda \to \infty$ limit are the second term of Eq. (66),

[15] In Eq. (68), $\left(D_\mu + ik_\mu\right)^2$ is a shorthand for $(D_\mu + ik_\mu)(D^\mu + ik^\mu)$.

[16] Note that this definition comes from the fact the metric in Euclidean space is $g_{\mu\nu} = -\delta_{\mu\nu}$.

[17] Note that e^{-k^2} can be factored out at the last line of Eq. (69) since it is proportional to the identity matrix which commutes with all the other elements.

and the exponential $e^{-\widetilde{k}/2}$. In the end we get

$$\mathcal{W} := \lim_{\Lambda\to\infty} \mathcal{W}_\Lambda = \int \frac{\mathrm{d}^4\widetilde{k}}{(2\pi)^4} e^{-\widetilde{k}^2} \left(\frac{e^2}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{72}$$

Then, from the Eq. (60) we arrive at the final result for the Jacobian:

$$\mathcal{J} = \exp\left\{-i\int \mathrm{d}^4x_E\,\epsilon(x)\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}. \tag{73}$$

Non-conservation of the axial current

Now we have all the information to show that the axial current J^μ_A is not conserved in the quantum theory. For this, we start with the analog of Eq. (21) in Euclidean space with the Dirac fermions:[18]

$$Z_E[\overline{K},K] = \int \mathcal{D}\overline{\psi}'\mathcal{D}\psi' \exp\left\{-S_E[\psi,\overline{\psi}] + \int \mathrm{d}^4x_E\,\left(\overline{K}\psi + K\overline{\psi}\right)\right\} \times\left[1 - i\int \mathrm{d}^4x_E\,\epsilon(x)\left(\partial_\mu J^\mu_A + i\overline{K}\psi + iK\overline{\psi}\right)\right], \tag{76}$$

where K and $\overline{K}$ are external sources for ψ and $\overline{\psi}$ respectively. From the Eq. (73) we know that the modified fermion measure reads

$$\int \mathcal{D}\overline{\psi}'\mathcal{D}\psi' = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{J}^2 = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left\{i\int \mathrm{d}^4x_E\,\epsilon(x)\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}. \tag{77}$$

By substituting this results into Eq. (76), expanding the exponential, and setting $K=0$ and $\overline{K}=0$ gives[19]

$$\left\langle \partial_\mu J^\mu_A\right\rangle = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{78}$$

We see that the axial current is no longer conserved, which means that the axial symmetry is anomalous. In the literature, this is known as the chiral anomaly or ABJ anomaly which is named after Adler, Bell, and Jackiw. In this example, the fermions transform under a U(1) gauge symmetry which is abelian. Therefore, this result is also called abelian anomaly.

The Anomaly with Non-abelian Gauge Theories

In the previous example the gauge symmetry under which the fermions transform was an abelian gauge symmetry. We will be eventually interested in QCD where the fermions transform under the fundamental

[18] On the second line of Eq. (76) the term $\partial_\mu J^\mu_A$ comes with the same sign and the factor of i even in the Euclidean space due to the fact that the path integral weight for fermions has the same form both in Minkowski and Euclidean spaces, namely

$$e^{iS} = \exp\left\{\int \mathrm{d}^4x\, i\,\overline{\psi}\not{D}\psi\right\}, \tag{74}$$

and

$$e^{-S_E} = \exp\left\{\mathrm{d}^4x_E\, i\,\overline{\psi}\not{D}_E\psi\right\}. \tag{75}$$

[19] Precisely speaking there should also be an expectation value on the RHS of Eq. (78). However, in anomaly calculations the gauge fields are taken to be background fields, hence they are classical. One can also take the gauge fields to be dynamical. The calculation and the results are not changed.

representation of $\mathrm{SU}(3)_C$ which is a non-abelian symmetry. At first, it might seem that we need to perform the whole anomaly calculation from scratch. However, it is fairly trivial to generalize the previous calculation for non-abelian gauge symmetries. We only need to modify Eq. (66) to

$$\not{D}^2 = D^2 - \frac{ig}{2}\gamma_E^\mu \gamma_E^\nu F_{\mu\nu}^a T^a, \tag{79}$$

where g is the gauge coupling constant, $\{T^a\}$ are the generators in the fundamental representation, and $F_{\mu\nu}^a$ is the gauge field strength. Recall that the main contribution to the anomaly comes from squaring the second term and taking the trace together with γ^5. Now we also need to take a trace over the group generators. So all the results in the abelian case can be applied to the non-abelian case by replacing

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \to \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^b \operatorname{Tr}\left(T^a T^b\right) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a, \tag{80}$$

where we have used the convention that the generators in the fundamental representation are normalized by $\operatorname{Tr}\left(T^a T^b\right) = \delta_{ab}/2$.

We can directly apply this to the massless QCD Lagrangian (1). By making an axial rotation given in Eq. (5) to a single quark the path integral measure changes by

$$\begin{aligned} \int \mathcal{D}\bar{q}\mathcal{D}q &\to \int \mathcal{D}\bar{q}\mathcal{D}q \exp\left\{ i\alpha \int \mathrm{d}^4x \, \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a \right\} \\ &= \int \mathcal{D}\bar{q}\mathcal{D}q \exp\left\{ i\alpha \int \mathrm{d}^4x \, \frac{g_s^2}{16\pi^2} G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu} \right\}. \end{aligned} \tag{81}$$

This modification corresponds to a change in the QCD Lagrangian given by

$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L} - \frac{\alpha g_s^2}{16\pi^2} G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}. \tag{82}$$

Note that this modification is purely a quantum effect and cannot be seen from a classical analysis.

But they are more surprises. It turns out that $G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu}$ is a total derivative! It can be written as

$$G_{\mu\nu}^a \widetilde{G}^{a\,\mu\nu} = \partial_\mu \epsilon^{\mu\alpha\beta\gamma}\left(A_\alpha^a G_{\beta\psi}^a - \frac{g_s}{3} f^{abc} A_\alpha^a A_\beta^b A_\gamma^c\right) =: \partial_\mu \mathcal{K}^\mu, \tag{83}$$

where $\mathcal{K}^\mu$ is the Chern-Simons current. So, at first sight it seems that it should not contribute to the equations of motion, and should be irrelevant. This is not true though. As we will see in the next lecture, the QCD has a complicated vacuum structure, and this term will play a vital role.

References

Bertlmann, Reinhold A. (2001). *Anomalies in Quantum Field Theory*. Revised ed. edition. Oxford: Oxford University Press. 584 pp. ISBN: 978-0-19-850762-8.

Bilal, Adel (Feb. 2008). "Lectures on Anomalies". In: arXiv: 0802.0634 [hep-th].

Tong, David (2018). *Gauge Theory*. URL: https://www.damtp.cam.ac.uk/user/tong/gaugetheory/gt.pdf.

A Grassmann numbers

In this section we will review the Grassmann numbers, also known as anti-commuting numbers or Grassmann variables.

A set of Grassmann numbers $\{\theta_i\}_{i=1}^n$ obey the anti-commutation relations:

$$\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0, \tag{84}$$

but add commutatively:

$$\theta_i + \theta_j = \theta_j + \theta_i. \tag{85}$$

They form a basis of the so-called Grassmann algebra $\mathfrak{G}$ over a field where the latter is usually taken as the complex numbers. This algebra contains an identity element 0 so that $\theta_i + 0 = \theta_i$. We can multiply a Grassmann number with a complex number and the result will be an element of the Grassmann algebra.

Let us consider the simplest case with $n = 1$. Since $\theta^2 = 0$, any function $f(\theta)$ of the Grassmann variable can be defined by its Taylor expansion:

$$f(\theta) = a + b\theta, \quad a, b \in \mathbb{C}, \tag{86}$$

where higher order terms are identically zero. Note that this also means that we can write any element of the Grassmann algebra as in Eq. (86).

In physics, the Grassmann numbers are used to express the fermionic fields. Since the path integral over them should give complex numbers, we need to find a way to define the integral of a Grassmann number to be a map from the Grassmann algebra to complex numbers. We also want the integration should have similar properties as the regular integration. In particular, the linearity

$$\int \mathrm{d}\theta\, (cf(\theta) + dg(\theta)) = c\int \mathrm{d}\theta\, f(\theta) + d\int \mathrm{d}\theta\, g(\theta), \tag{87}$$

and invariance under a constant shift of the parameter,

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, f(\theta + \eta). \tag{88}$$

The combination of these requirement implies that

$$\int d\theta = 0 \tag{89}$$

The value of $\int d\theta\,\theta$ needs to be chosen, and the convention is to set it to unity. In summary, the integration of Grassmann numbers are defined by

$$\int d\theta \equiv 0, \quad \int d\theta\,\theta \equiv 1. \tag{90}$$

This is known as the Berezin integral. This implies that the integral of an element of the Grassmann algebra like in Eq. (86) is *defined* to be

$$\int d\theta\, f(\theta) = \int d\theta\,(a + b\theta) \equiv b. \tag{91}$$

We can easily generalize all these definitions to general n. Now, a general function $f(\{\theta_i\})$ can be written as

$$f(\{\theta\}) = a + b_i\theta_i + \frac{1}{2}c_{i_1,i_2}\theta_{i_1}\theta_{i_2} + \cdots + \frac{1}{n!}d_{i_1\cdots i_n}\theta_{i_1}\cdots\theta_{i_n}, \tag{92}$$

where the indices are implicitly summed. The coefficients should be anti-symmetric. Thus, the coefficient of the last term can be written as

$$d_{i_1\cdots i_n} = \epsilon_{i_1\cdots i_n} d, \tag{93}$$

where $\epsilon_{i_1\cdots i_n}$ is the Levi-Civita symbol with $\epsilon_{1\cdots n} = +1$. The integral of $f(\{\theta\})$ is defined by[20]

$$\int d^n\theta\, f(\{\theta\}) \equiv d. \tag{94}$$

[20] Alternatively, we can take $d^n\theta = d\theta_n \cdots d\theta_1$, treat the diffentials as anti-commuting, i.e. $\{d\theta_i, d\theta_j\} = 0 = \{d\theta_i, \theta_j\}$, and define $d\theta_i = 0$ and $d\theta_i\,\theta_j = \delta_{ij}$ to obtain Eq. (94).

This equation tells us something important. Consider we make a change of variables from $\{\theta_i\}$ to $\{\theta'_i\}$ via

$$\theta_i = X_{ij}\theta'_j, \tag{95}$$

where X is a matrix of commuting numbers. In this new basis, Eq. (92) becomes

$$f(\{\theta'\}) = a + b_i X_{ij}\theta'_j + \cdots + \frac{1}{n!}d\,\underbrace{\epsilon_{i_1\cdots i_n}X_{i_1j_1}\cdots X_{i_nj_n}}_{=\det X}\theta'_{i_1}\cdots\theta'_{i_n}. \tag{96}$$

Then, From Eq. (94) we learn that

$$\int d^n\theta\, f(\{\theta\}) = (\det X)^{-1}\int d^n\theta'\, f(\{\theta'\}). \tag{97}$$

Note that this Jacobian comes with an inverse power compared to the case when the integration variables are usual numbers.