Lecture 3: Chiral Lagrangians

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In the last lecture, we have seen that the chiral symmetry breaking pattern in the QCD is $\mathsf{SU}(N)_L \otimes \mathsf{SU}(N)_R \longrightarrow \mathsf{SU}(N)_V$ where $N = 2$ if we take only up and down quarks as massless, and $N = 3$ with the inclusion of the strange quark. The Goldstone Theorem implies there should be three and eight massless degrees of freedom for $N = 2$ and $N = 3$ respectively. The lightest three degrees of freedom are the pions $\pi^0, \pi^\pm$, while the remaining ones are kaons $\overline{K}^0$, $\underline{K}_0$, $K^\pm$ and the eta meson η. Since the chiral symmetry is only approximate in the QCD, and explicitly broken by the non-zero quark masses, the mesons are pseudo-Nambu-Goldstone Bosons (pNGBs).

In this lecture, we will see how to construct an *effective* Lagrangian for the mesons. As we will see, we will achieve this by only utilizing the symmetry breaking pattern.

1 Framework

Let us briefly remember the terminology that we have introduced so far, and also introduce some object that will be useful. We grouped the left-handed and right-handed components of the quarks and the anti-quarks into the vectors defined by[1]

$$\mathbf{q}_{L,R} := \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}, \quad \widetilde{\mathbf{q}}_{L,R} := \begin{pmatrix} u^\dagger_{L,R} \\ d^\dagger_{L,R} \end{pmatrix}. \tag{1}$$

[1] We will consider only two massless quarks for simplicity but the whole discussion can be straightforwardly generalized to the case with three massless quarks.

For notational simplicity, in this lecture we will denote a $\mathsf{SU}(2)_L$ transformation by L, and a $\mathsf{SU}(2)_R$ transformation by R. The transformations of the quark vectors are given explicitly by

$$\mathbf{q}_L \mapsto L\mathbf{q}_L, \quad q_{L,i} \mapsto L_{ij} q_{L,j} \tag{2a}$$
$$\mathbf{q}_R \mapsto R\mathbf{q}_R, \quad q_{R,i} \mapsto R_{ij} q_{R,j} \tag{2b}$$
$$\widetilde{\mathbf{q}}_L \mapsto L^*\widetilde{\mathbf{q}}_L, \quad q^\dagger_{L,i} \mapsto L^*_{ij} q^\dagger_{L,j} \tag{2c}$$
$$\widetilde{\mathbf{q}}_R \mapsto R^*\widetilde{\mathbf{q}}_R, \quad q^\dagger_{R,i} \mapsto R^*_{ij} q^\dagger_{R,j}, \tag{2d}$$

where $q_{L,i} := (\mathbf{q}_L)_i$ and similarly for other vector components, and $*$ denotes the complex conjugate. The combined symmetry transformation $G := \mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R \longrightarrow \mathsf{SU}(2)_V$ can alternatively be expressed as

$$G : \mathbf{q} \mapsto \exp\{i(\theta^a T^a + \gamma_5 \beta^a T^a)\}\mathbf{q}, \tag{3}$$

where $\mathbf{q}$ is a vector of Dirac spinors, i.e.

$$\mathbf{q} := \begin{pmatrix} u \\ d \end{pmatrix}, \quad \overline{\mathbf{q}} := \begin{pmatrix} \overline{u} \\ \overline{d} \end{pmatrix}, \tag{4}$$

and $\{T^a\}$ are the $\mathsf{SU}(2)$ generators. Note that in this expression the generators act on the vector $\mathbf{q} = \begin{pmatrix} u \\ d \end{pmatrix}$ while the gamma matrix γ_5 acts on the Dirac spinors u and d.

From Eq. (3) we can see that they are two sets of transformations:

1. The set of transformations parametrized by θ^a with $\beta^a = 0$.
2. The set of transformations parametrized by β^a with θ^a.

The first set of transformations rotate the left-handed and right-handed spinors identically, i.e. $L = R$. This is precisely the vector subgroup $\mathsf{SU}(2)_V$ which remains unbroken even after the chiral phase transitions. The second set of transformations rotate the left-handed and right-handed spinors with opposite phases, i.e. $L = R^\dagger$. These are called axial rotations and these correspond to the symmetries broken by the quark condansate.

2 Fluctuations around the vacuum

Let us recall the *Mexican hat* potential that we have studied in the first lecture, see Figure 1. There we had a complex scalar field enjoying a $\mathsf{U}(1)$ global symmetry which is spontaneously broken. The vacuum was infinitely degenerate, and the Goldstone boson corresponds to the angular excitations around a choice of vacuum.

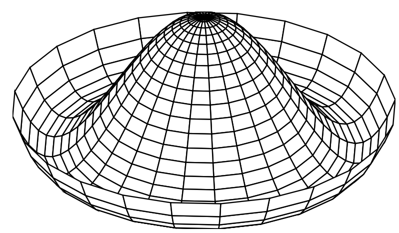

Figure 1: A plot of the Mexican hat potential with a broken phase.

We are seeking to find a similar procedure for the chiral symmetry breaking. For this, the first step should be defining the vacuum. We already know that the chiral symmetry breaking corresponds to

$$\langle \overline{u} u \rangle = \left\langle \overline{d} d \right\rangle \sim \Lambda^3_{\mathrm{QCD}}. \tag{5}$$

In terms of the quark vector (5) this can be written compactly as

$$\left\langle \overline{q}_j q_i \right\rangle = \left\langle q^\dagger_{R,j} q_{L,i} \right\rangle + \left\langle q^\dagger_{L,j} q_i \right\rangle \propto \delta_{ij}. \tag{6}$$

Since the second term is the hermitian conjugate of the first term when $i = j$ we can write

$$\left\langle \overline{q}_j q_i \right\rangle = \left\langle q^\dagger_{R,j} q_{L,i} \right\rangle + \text{h.c.} \propto \delta_{ij} \tag{7}$$

We now define a composite field $\Sigma(x)$ whose matrix elements are defined by

$$\Sigma_{ij}(x) := q^\dagger_{R,j}(x) q_{L,i}(x). \tag{8}$$

Under a $G = \mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$ transformation with $L \in \mathsf{SU}(2)_L$ and $R \in \mathsf{SU}(2)_L$, the object Σ transforms as

$$G : \Sigma \mapsto L\,\Sigma\,R^\dagger, \quad G : \Sigma \mapsto R\,\Sigma^\dagger\,L^\dagger. \tag{9}$$

This results implies that Σ transforms as a bifundamental with respect to $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$. The Eq. (7) tells us that the VEV of this field should satisfy

$$\left\langle \Sigma_{ij} \right\rangle + \left\langle \Sigma^\dagger_{ij} \right\rangle \propto \delta_{ij}. \tag{10}$$

We see that this expression remains invariant under a vector transformation with $L = R$. This is another statement that the vacuum is still invariant under the $\mathsf{SU}(2)_V$ subgroup. Moreover, Eq. (10) immediately tells us that the $\langle \Sigma \rangle$ should be hermitian, so it should be proportional to the identity matrix:[2]

[2] From Eq. (10) we learn that the real parts of the diagonal components of $\langle \Sigma \rangle$ should be identical so in general their imaginary parts might differ. However, we can make them identical by a vector transformation $\Sigma \mapsto L \Sigma L^{-1}$ which leaves the vacuum state invariant.

$$\langle \Sigma \rangle_{ij} \propto \delta_{ij} \quad \Rightarrow \quad \Sigma_0 := \langle \Sigma \rangle \propto 1. \tag{11}$$

The fluctuations of $\Sigma(x)$ around the VEV Σ_0 correspond to the Goldstone bosons. Remember that the action of an unbroken symmetry does not affect the VEV while broken symmetries shift the VEV around the vacuum manifold. To identify the Goldstones, we need the latter. In the end, the procedure to obtain the Goldstones can be summarized as

1. Identify a convenient VEV, i.e. Σ_0.
2. Act on that VEV with the broken group elements.
3. Promote transformation parameters to fields and identify them with the Goldstones.

For the chiral symmetry breaking, the broken symmetries are those with $L = R^\dagger$. Then, by acting on the VEV, i.e. the identity we obtain[3]

[3] Without loss of generality we can take Σ_0 to be *equal* to the identity matrix by absorbing any dimensionful constant in Eq. (10) into the definition of $\Sigma(x)$.

$$\exp\{i\epsilon^a T^a\} \cdot 1 \cdot \exp\{i\epsilon^a T^a\} = \exp\{2i\epsilon^a T^a\} \tag{12}$$

We now promote the parameters $\{\epsilon^a\}$ to Goldstone fields $\{\pi^a(x)\}$ and introduce a dimensionful quantity f so that the Goldstones have mass dimension one. This way we define the field $\Sigma(x)$

$$\Sigma(x) := \exp\left\{2i\frac{\pi^a(x)}{f}T^a\right\}, \tag{13}$$

which transforms as a bifundamental under the $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$ group:

$$\Sigma(x) \mapsto L\,\Sigma(x)\,R^\dagger. \tag{14}$$

At this point it will also be useful to write $\Sigma(x)$ explicitly:

$$\Sigma(x) = \exp\left\{\frac{i}{f}\begin{pmatrix} \pi^3 & \pi^1 - i\pi^2 \\ \pi^1 + i\pi^2 & -\pi^3 \end{pmatrix}\right\}. \tag{15}$$

The physical states π^0 and $\pi^\pm$ are written in terms of $\{\pi^a\}$ as

$$\pi^0 = \pi^3, \quad \pi^\pm = \frac{1}{\sqrt{2}}\left(\pi^1 \pm \pi^2\right). \tag{16}$$

3 How pions transform?

The transformation rules of the pions can be determined by the transformation of $\Sigma(x)$. Acting with an element V of the unbroken $\mathsf{SU}(2)_V$ symmetry yields

$$\begin{aligned} V\,\Sigma(x)\,V^\dagger =& V\exp\left\{2i\frac{\pi^a}{f}T^a\right\}V^\dagger \\ =& V\left(1+2i\frac{\pi^a}{f}T^a+\frac{1}{2}(2i)^2\frac{\pi^a\pi^b}{f^2}T^aT^b+\cdots\right)V^\dagger \\ =& \exp\left\{V\left(2i\frac{\pi^a}{f}T^a\right)V^\dagger\right\}, \end{aligned} \tag{17}$$

where the last identity follows from inserting bunch of identities $V^\dagger V = 1$ to the matrix expansion of the exponential. This result tells us that the Goldstones transform under the unbroken $\mathsf{SU}(2)_V$ symmetry as

$$\mathsf{SU}(2)_V : \pi^a T^a \mapsto V\,\pi^a T^a\,V^\dagger. \tag{18}$$

This is a linear transformation so the conclusion is that the Goldstones transform under the unbroken symmetry in a linear way.

Let us see what happens if we act on the vacuum with the broken generators $L = R^\dagger =: A$. In this case we cannot put identities as in the previous case to obtain a simple expression for the transformed fields. All we can say that an axial transformation sends $\pi^a T^a$ to $\pi'^a T^a$ where the latter is defined through

$$A\,\Sigma(x)\,A =: \exp\left\{2i\frac{\pi'^a(x)}{f}T^a\right\}. \tag{19}$$

This is a non-linear transformation so one says that the Goldstones transform non-linearly under the broken symmetry. By expanding both sides to leading order we obtain

$$A : \pi^a(x)T^a \mapsto \pi^a(x)T^a + f\alpha^a T^a. \tag{20}$$

The second term is just a constant so an axial transformation is realized as a shift symmetry for the Goldstones. This is precisely the symmetry which makes the Goldstones massless.

4 Lagrangian description

Next, we want to construct a Lagrangian for $\Sigma(x)$ and $\Sigma^\dagger(x)$ that respects the symmetries of the full $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$ group. Note that

$$\Sigma \mapsto L\,\Sigma\,R^\dagger \quad \Rightarrow \quad \Sigma^\dagger \mapsto R\,\Sigma\,L^\dagger. \tag{21}$$

The most general invariant term with no derivatives takes the form

$$\mathrm{Tr}\left[\Sigma^\dagger\Sigma\ldots\Sigma^\dagger\Sigma\right]. \tag{22}$$

However $\Sigma^\dagger\Sigma$ is proportional to the identity matrix so all these terms are just constants and independent of the Goldstones. All the terms with an odd number of derivatives are absent due to the Lorentz invariance. Thus, the most general Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots, \tag{23}$$

where $\mathcal{L}_2$ and $\mathcal{L}_4$ denotes the terms with two and four derivatives. There is a single term with two derivatives which is

$$\mathcal{L}_2 = b\,\mathrm{Tr}\left[\left(\partial^\mu\Sigma^\dagger(x)\right)\left(\partial_\mu\Sigma(x)\right)\right], \tag{24}$$

where b is a constant which we shall fix shortly. We can know substitute Eq. (13) into this and perform the expansion. This way we get

$$\begin{aligned}\mathcal{L}_2 &= \frac{4b}{f^2}\left(\partial^\mu\pi^a(x)\right)\left(\partial_\mu\pi^a(x)\right)\mathrm{Tr}\left[T^aT^b\right] + \cdots, \\ &= \frac{2b}{f^2}\left(\partial^\mu\pi^a(x)\right)\left(\partial_\mu\pi^a(x)\right) + \cdots,\end{aligned} \tag{25}$$

where we have used the orthogonality condition for the generators:

$$\mathrm{Tr}\left[T^aT^b\right] = \frac{1}{2}\delta^{ab}. \tag{26}$$

To canonically normalize the pion fields we need fix $b = f^2/4$ so the final form of $\mathcal{L}_2$ is

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{Tr}\left[\left(\partial^\mu\Sigma^\dagger(x)\right)\left(\partial_\mu\Sigma(x)\right)\right]. \tag{27}$$

This Lagrangian determines both the kinetic terms for the pions, and also includes terms denoting the multi-pion scattering amplitudes.

5 Explicit symmetry breaking

The $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ symmetry is explicitly broken by the quark mass terms:[4]

$$\Delta\mathcal{L} = -\sum_q m_q\bar{q}q = -\sum_q m_q\left(q_L^\dagger q_R\right) + \text{h.c.} \tag{28}$$

By introducing the quark mass matrix

$$M := \begin{pmatrix} m_u & \\ & m_d \end{pmatrix}, \tag{29}$$

[4] "h.c." denotes hermitian conjugate.

we can write this as

$$\Delta\mathcal{L} = -\widetilde{\mathbf{q}}_L \cdot M\, \mathbf{q}_R. \tag{30}$$

Under a $\mathsf{SU}(3)_L \otimes \mathsf{SU}(3)_R$ transformation

$$\Delta\mathcal{L} \to -\widetilde{\mathbf{q}}_L \cdot L^\dagger\, M\, R\, \mathbf{q}_R \neq \Delta\mathcal{L}, \tag{31}$$

so the Lagrangian is no longer invariant, and one says that the quark masses break the chiral symmetry explicitly. However, if this breaking is small, we can use a trick known as the spurion analysis to incorporate the quark masses into the Chiral Lagrangian.

Let us promote the quark mass matrix M to a field that transforms under a $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R$ transformation such that $\Delta\mathcal{L}$ remains invariant. This requires

$$M \mapsto L\, M\, R^\dagger. \tag{32}$$

Then we can construct the Chiral Lagrangian out of Σ and M such that under the transformations of these fields given by Eq. (9) and Eq. (32) respectively, it remains invariant. The lowest order term that comes from the inclusion of quark masses is given by

$$\mathcal{L}_m = \frac{1}{4}\mu f^2 \Big[\mathrm{Tr}\Big(\Sigma^\dagger M\Big) + \mathrm{Tr}\Big(M^\dagger \Sigma\Big)\Big], \tag{33}$$

where μ is a constant with mass dimension one, and the factor f^2 is extracted for later convenience. We can know plug Eq. (13) and Eq. (29) into this equation and perform the expansion. This way, we get

$$\mathcal{L}_m = \text{constant} - \frac{\mu}{2}(m_u + m_d)\pi^a\pi^a. \tag{34}$$

Unfortunately, this does not give us the meson masses directly due to the unknown dimensionful constant μ. However, by generalizing the discussion in this lecture to the spontaneous breaking of $\mathsf{SU}(3)_L \otimes \mathsf{SU}(3)_R$ one can eliminate the constant μ and express the quark mass ratios in terms of the meson masses:

$$\frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 0.55 \tag{35}$$

$$\frac{m_s}{m_d} = \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20.1, \tag{36}$$

and also the well-known Gell-Mann-Okubo formula:

$$\underbrace{4M_{K^0}^2}_{\simeq 992\,\mathrm{GeV}^2} = \underbrace{3M_\eta^2 + M_\pi^2}_{\simeq 919\,\mathrm{GeV}^2}. \tag{37}$$

We see that even in the lowest order prediction of the Chiral perturbation theory is roughly consistent with the known meson masses that are provided in the handout for Lecture 2b. However, we should note that the quark masses in these relations are whatever renormalized masses appear in the Lagrangian (28). They are not the renormalized $\overline{\mathrm{MS}}$ masses that we have tabulated in Lecture 2b.

6 The pion decay constant

We close this lecture by briefly stating how the dimensionful constant f is determined. In the Standard Model, the electroweak interactions are given by the gauge group $\mathrm{SU}(2)_L$ under the left-handed components of the up and down quarks form the first generation quark doublet.

$$Q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix}. \tag{38}$$

But this doublet coincides with the first two elements of the vector $\mathbf{q}_L$ on which a $\mathrm{SU}(3)_L$ transformation acts. The terminology is that the electroweak group is weakly gauged with respect to the low energy QCD. Here weakly means that the gauge couplings are perturbative in all energy scales of interest.

This weakly gauging of the electroweak groups allows us to incorporate the weak interactions into the chiral Lagrangians. Since we are at the low energies, it is appropriate to use the 4-Fermi theory which is a low-energy effective theory of the weak interactions. This way it is possible to calculate the rates for the electroweak decays of the charged mesons $\pi^{\pm}$. The most important is the decay rate for $\pi^+ \to \mu^+ \nu_\mu$ which is

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 f^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2, \tag{39}$$

where $G_F = 1.16 \times 10^{-5}\,\mathrm{GeV}^{-2}$, m_μ is the muon mass, and m_π is the charged pion mass. Here f is the same quantity that we have used when defining the Goldstones. Using the measured value for the muon lifetime

$$\tau = \Gamma^{-1} = 2.6 \times 10^{-8}\,\mathrm{s}, \tag{40}$$

one finds f as

$$f = 92\,\mathrm{MeV}. \tag{41}$$

For this reason, f is called the pion decay constant and usually denoted by f_π or F_π.