Lecture 2b: Chiral Symmetry Breaking in QCD

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In this lecture, we shall give an introduction to the Chiral Symmetry Breaking in QCD. In particular, we describe the symmetry breaking pattern which we will use to construct the Chiral QCD Lagrangian in the next lecture.

1 The QCD Lagrangian

We start by stating the QCD Lagrangian in the basis where the mass matrix is diagonal:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_q \left(i\,\bar{q} \not{D} q - m_q \bar{q} q \right). \tag{1}$$

As we shall see later, there is also the θ-term which is proportional to $\tilde{G}^a_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma} G^{a\,\rho\sigma}$, but we will ignore it for now. Let us remember the notations in this Lagrangian:

- q is a Dirac fermion representing one of the quarks:[1]
- $\bar{q}$ is the Dirac adjoint of q given by $\bar{q} := q^\dagger \gamma^0$ with $\{\gamma^\mu\}$ being the Gamma matrices.[2]
- $G^a_{\mu\nu}$ is the gluon field strength given by

  $$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu, \tag{2}$$

  where g_s gauge coupling constant, i.e. the analog of the fine structure constant in QED. The eight gluons are represented by the gauge fields $\left\{ A^a_\mu \right\}_{a=1}^{8}$ where the super-script a labels them. The structure constants enter to this definition due to the fact that the gluons, being gauge fields, transform under the adjoint representation of the $\mathrm{SU}(3)_C$ where the representations of the group generators take the form $\left(T^a_{\text{adj}} \right)^{bc} = -i f^{abc}$.
- The kinetic term for the quarks is

$$\begin{aligned} \bar{q} \not{D} q &= \bar{q} \gamma^\mu D_\mu q \\ &= \bar{q}_i \gamma^\mu \underbrace{\left(\delta_{ij} \partial_\mu - i g_s A^a_\mu T^a_{ij} \right)}_{=(D_\mu)_{ij}} q_j, \end{aligned} \tag{3}$$

[1] "u" for up, "d" for down, "s" for strange, "c" for charm, "b" for bottom, and "t" for top.

[2] Our convention is to take the γ-matrices in the Weyl representation. Explicitly

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where $\{\sigma^i\}$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where i and j are the color indices of the quarks, i.e. red, blue, and green. Here $\{T^a\}$ are the generators of $\mathsf{SU}(3)_C$ in the fundamental representation. This is because the quarks, being matter fields, transform under the fundamentel representation. The generators are given by $T^a = \lambda^a/2$ where $\{\lambda^a\}$ are the Gell-Mann matrices.[3]

[3] See the handout for the Lecture 2a for the explicit expressions.

2 Symmetries of the QCD Lagrangian with massless quarks

Now we discuss the symmetries of the Lagrangian. It is easy to see that there is a $\mathsf{U}(1)$ symmetry which simply rotates the phases of the quarks by the same amount

$$\mathsf{U}(1)_V : q \mapsto e^{i\alpha} q. \tag{4}$$

This is called the vector $U(1)$ symmetry.

Without any approximation, this is all we have. But this is not very useful since, as we shall see later, this symmetry remains unbroken after the QCD phase transition. To make further progress, we observe that the Lagrangian contains a much larger symmetry group when the quark masses are set to zero. To see these symmetries more clearly, we recall that a Dirac fermion in the Weyl basis can be written in terms of a left-handed q_L and a right-handed Weyl spinor q_R as[4]

[4] Remember that left-handed and right-handed spinors transform under the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of the Lorentz group.

$$q = \begin{pmatrix} q_L \\ q_R \end{pmatrix}. \tag{5}$$

Then, a simple calculation shows that

$$i\,\bar{q}\not{D}q = i\left(q_L^\dagger \bar{\sigma}^\mu D_\mu q_L + q_R^\dagger \sigma^\mu D_\mu \sigma q_R\right), \tag{6}$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. We can group different quarks and anti-quarks into the vectors

$$\mathbf{q}_{L,R} := \begin{pmatrix} u_{L,R} \\ d_{L,R} \\ \vdots \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{q}}_{L,R} := \begin{pmatrix} u_{L,R}^\dagger \\ d_{L,R}^\dagger \\ \vdots \end{pmatrix}. \tag{7}$$

Then by summing over the quarks we obtain

$$\sum_q i\,\bar{q}\not{D}q = i\left(\tilde{\mathbf{q}}_L \cdot \bar{\sigma}^\mu D_\mu \mathbf{q}_L + \tilde{\mathbf{q}}_R \cdot \sigma^\mu D_\mu \mathbf{q}_R\right). \tag{8}$$

Now consider the following transformations acting on the quark vectors:

$$U_L : \mathbf{q}_L \mapsto U_L \mathbf{q}_L \quad , \quad U_R : \mathbf{q}_R \mapsto U_R \mathbf{q}_R. \tag{9}$$

This is a symmetry of the Lagrangian in the massless limit if U_L and U_R are unitary matrices, i.e. $U_L^\dagger U_L = 1$ and $U_R^\dagger U_R = 1$. If they are

N massless quarks, these transformations are $N \times N$ unitary matrices and they form the Special Unitary Group $\mathsf{SU}(N)$. We will denote these symmetries as $\mathsf{SU}(N)_L$ and $\mathsf{SU}(N)_R$ where $U_L \in \mathsf{SU}(N)_L$ and $U_R \in \mathsf{SU}(N)_R$. These symmetries are called left chiral and right chiral symmetry respectively.

Another symmetry of the Lagrangian with massless quarks is the axial symmetry $\mathsf{U}(1)_A$ given by

$$U(1)_A : q \mapsto e^{i\theta\gamma_5} q, \tag{10}$$

where

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in the Weyl basis.} \tag{11}$$

However, this is not a true symmetry of the Lagrangian. It is broken by quantum effects called anomalies. We will discuss anomalies extensively in the next lectures.

3 Chiral Symmetry Breaking Pattern

The QCD phase transition is realized as a result of quark bilinears $\bar{q}q$ obtain a non-zero vacuum expectation value

$$\langle \bar{q}q \rangle \sim \Lambda^3_{\text{QCD}}, \quad \Lambda \sim 300\,\text{MeV}. \tag{12}$$

So far this has not been proved from a first principle calculation in QCD, but it is consistent with the spectrum of hadrons. We now discuss which symmetries are broken by this expectation value.

The vector symmetry $\mathsf{U}(1)_V$ remains unbroken. Its conserved Noether charge is the baryon number. Even though the baryon number is an exact symmetry of the QCD even with massive quarks, it also become anamolous once the weak interactions are included.

The axial symmetry $\mathsf{U}(1)_A$ is broken by the expectation value. However, since it wasn't a real symmetry to begin with, it plays no further role.

We now discuss the most important symmetry which is the combined $\mathsf{SU}(N)_L \otimes \mathsf{SU}(N)_R$ symmetry. By expanding (12) in terms of Weyl fermions we get

$$\langle \bar{q}q \rangle = \left\langle q_L^\dagger q_R + q_R^\dagger q_L \right\rangle, \tag{13}$$

and summing over the quarks yields

$$\sum_q \langle \bar{q}q \rangle = \langle \widetilde{\mathbf{q}}_L \cdot \mathbf{q}_R + \widetilde{\mathbf{q}}_R \cdot \mathbf{q}_L \rangle \neq 0. \tag{14}$$

Under the transformation (9), this is transformed to

$$\sum_q \langle \bar{q}q \rangle \mapsto \left\langle \widetilde{\mathbf{q}}_L \cdot U_L^\dagger U_R \mathbf{q}_R \right\rangle + \left\langle \widetilde{\mathbf{q}}_R \cdot U_R^\dagger U_L \mathbf{q}_L \right\rangle. \tag{15}$$

We see that the vacuum stays invariant if

$$U_L^\dagger U_R = 1 \quad \text{and} \quad U_R^\dagger U_L = 1. \tag{16}$$

This means that the vacuum state is not invariant anymore under *independent* $\mathsf{SU}(N)_L$ and $\mathsf{SU}(N)_R$ transformations. However, it stays invariant if these are related to each other via (16). The subgroup of $\mathsf{SU}(N)_L \otimes \mathsf{SU}(N)_L$ under which the vacuum is invariant is another $\mathsf{SU}(N)$ group[5] called the vector $\mathsf{SU}(N)_V$. So the symmetry breaking pattern of QCD with N massless quarks is

[5] The easiest way to see this is by looking Eq. (16). One is free to choose either a $\mathsf{SU}(N)_L$ or $\mathsf{SU}(N)_R$ rotation, but the rest is fixed.

$$\mathsf{SU}(N)_L \otimes \mathsf{SU}(N)_R \longrightarrow \mathsf{SU}(N)_V \tag{17}$$

Since each $\mathsf{SU}(N)$ has $N^2 - 1$ generators, the number of broken generators is

$$\underbrace{(N^2-1)}_{\mathsf{SU}(N)_L} + \underbrace{(N^2-1)}_{\mathsf{SU}(N)_L} - \underbrace{(N^2-1)}_{\mathsf{SU}(N)_V} = N^2 - 1. \tag{18}$$

Then using the Goldstone theorem, we conclude that the chiral symmetry breaking of QCD with N massless quarks should produce $N^2 - 1$ NGBs.

4 QCD with Massive Quarks

Of course we know that setting all quark masses to zero is an idealization, and in reality all quarks are massive. However, we are interested in an effective description of QCD at energies $E \ll \Lambda_{\text{QCD}} \sim 300\,\text{MeV}$. So if at least some of the quark masses are small compared to Λ_{QCD} we can treat them as *perturbations*. In this case, we can still employ the techniques of spontaneous symmetry breaking to tell something about the low energy theory.

Type	Mass
up	2.16 MeV
down	4.67 MeV
strange	93.4 MeV
charm	1.27 GeV
bottom	4.18 GeV
top	172.69 GeV

Table 1: Quark masses according to the Particle Data Group (PDG).

In Table 1, we list the six quark masses[6] according to the Particle Data Group (PDG).[7] We see that three of them, up, down, and strange, are lighter than the QCD scale, while two of them, up and down, are particularly light. The rest of them are much heavier so their masses cannot be treated as perturbations.

[6] The masses for up, down, and strange quarks are the $\overline{\text{MS}}$ masses where the renormalization scale is set to $\mu = 2\,\text{GeV}$. Check the relevant section in PDG on how to define the masses for the heavier quarks.

[7] R. L. Workman et al. (2022). In: *PTEP* 2022, p. 083C01.

With the inclusion of the quark masses, the chiral symmetries are no longer exact symmetries. But, if the quark masses can be treated as perturbations, then there is an approximate chiral symmetry. The resulting Goldstone bosons won't be massless anymore, but they are still light compared to the cutoff of our effective theory. In this case, the light but massive degrees of freedom are called pseudo Nambu-Goldstone bosons (pNGBs).

Name	**Symbol**	**Mass [MeV]**
Neutral Pion	π^0	135
Charged Pions	π^+, π^-	140
Charged Kaons	K^+, K^-	494
Neutral Kaons	$\underline{K}_0, \overline{K}^0$	498
Eta meson	η	548
Eta prime meson	η'	958
$\vdots$	$\vdots$	$\vdots$

Table 2: Lightest meson masses according to the Particle Data Group (PDG).

The light degrees of freedom that arise due to the spontaneous breaking of approximate chiral symmetry are the mesons which are bound states of a quark and an anti-quark. We list the lightest of them in Table 2. We know investigate whether these masses are consisten with what we expect from the symmetry breaking.

Two massless quarks

If we take only the up and down quarks as massless and integrate out the rest, then the symmetry breaking pattern is $\mathsf{SU}(2)_L \otimes \mathsf{SU}(2)_R \longrightarrow \mathsf{SU}(2)_V$ and we get $2^2 - 1 = 3$ pNGBs. We can match these with the pions $\pi^0, \pi^\pm$ since they are much lighter compared to the other mesons.

Three massless quarks

Now if we also include the strange quark, then the symmetry breaking pattern is $\mathsf{SU}(3)_L \otimes \mathsf{SU}(3)_R \longrightarrow \mathsf{SU}(3)_V$ and we get $3^2 - 1 = 8$ pNGBs. Three of them are again the pions. Four of the remaining five bosons are the kaons $\underline{K}_0, \overline{K}^0, K^\pm$, and the eta meson η. We also see that these are lighter compared to the next boson in the list which is the eta prime meson η'.

5 The $\mathsf{U}(1)_A$ Problem

In the 70s, people were not aware of the fact that the axial $\mathsf{U}(1)_A$ symmetry was anomalous. Since this symmetry is also spontaneously broken by the QCD condensate, it was expected that there should also be a light state corresponding to the pNGB of the $\mathrm{U}(1)_A$ symmetry. It has also been calculated there should be an upper bound on this boson's mass given by [8]

$$m^2 \leq 3m_\pi^2. \tag{19}$$

[8] Steven Weinberg (June 1975). In: *Phys. Rev. D* 11 (12), pp. 3583–3593.

But they was no such a state and people were wondering what happened to this boson. This was called the $\mathrm{U}(1)_A$ problem or the missing meson problem.

The resolution of the problem lies of course in the fact that the axial symmetry is anomalous. We will revisit this once we study the anamalies.

References

Weinberg, Steven (June 1975). "The U(1) problem". In: *Phys. Rev. D* 11 (12), pp. 3583–3593. URL: https://link.aps.org/doi/10.1103/PhysRevD.11.3583.

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