Lecture 2a: Lie Groups and Lie Algebras

Cem Eröncel

cem.eroncel@.itu.edu.tr

Lecture date: February 28, 2023

Last update: March 3, 2023

This handout provides a brief review of Lie groups and Lie algebras that we will use in later parts of the course.

1 Lie groups and Lie algebras

Let us recall the formal definition of a group:

Definition: A group is a set G together with a binary operation on G, denoted by $\bullet$, which takes any two group elements $a, b \in G$ to form another group element $a \bullet b \in G$, and satisfies the group axioms:

- Associativity: $a \bullet (b \bullet c) = (a \bullet b) \bullet c$, $\forall a, b, c \in G$,
- Existence of identity: $\forall a \in G \quad \exists e \in G \; : \; a \bullet e = e \bullet a = e$,
- Existence of inverse: $\forall a \in G \quad \exists a^{-1} \in G \; : \; a \bullet a^{-1} = a^{-1} \bullet a = e$.

The element e is called the identity while a^{-1} is the inverse of a.

It is not hard to convinve ourselves that the symmetry transformations form a group where the binary operation is simply the combination of two transformations. We will be mainly interested in the transformations that can be continuously connected to the identity. These transformations form a Lie group which is a group but at the same time a diffentiable manifold.

Being a differentiable manifold, the Lie group G has a tangent space $\mathfrak{g}$ attached to its identity element $e \in G$. This tangent space is a vector space spanned by the basis $\{T^a\}_{a=1}^{\dim G}$ where $\dim G$ is the dimension of the Lie group G. The set of basis vectors T^a are called the *group generators*. At the same time $\mathfrak{g}$ forms a Lie algebra:

Definition: A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear map called Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \quad , \quad (A, B) \mapsto [A, B] \tag{1}$$

that satisfies

- Anticommutativity: $[A, B] = -[B, A]$,
- Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

The definition of the Lie algebra, particularly Eq. (5), implies that we can write

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{2}$$

where f^{abc} are called structure constants. The group is called abelian if $f^{abc} = 0$, and non-abelian otherwise. The Jacobi identity implies that the structure constants should obey the relation

$$f^{abd} f^{cde} + f^{bcd} f^{ade} + f^{cad} f^{bde} = 0. \tag{3}$$

At this point the symbol $[\cdot,\cdot]$ should be understood as an abstract symbol denoting the Lie bracket. Soon, we will *represent* the group generators as matrices and then $[\cdot,\cdot]$ will imply the usual matrix commutation.

Definition: An ideal is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ satisfying

$$[g, h] \in \mathfrak{h} \quad \text{for any} \quad g \in \mathfrak{g} \quad \text{and} \quad h \in \mathfrak{h}. \tag{4}$$

A simple Lie algebra has no non-trivial ideals. If an algebra can be written as the direct sum of simple Lie algebras that it is called semi-simple. The gauge group of the Standard Model is

$$\mathsf{SU}(3)_c \otimes \mathsf{SU}(2)_{EW} \otimes \mathsf{U}(1)_Y$$

where the subscripts c, EW, and Y is for color, electro-weak, and hypercharge respectively. It's Lie algebra is given by

$$\mathfrak{su}(3)_c \oplus \mathfrak{su}(2)_{EW} \oplus \mathfrak{u}(1)_Y,$$

so the Standard Model Lie algebra is semi-simple. This is no surprise because of the following remarkable theorems:

Theorem (Cartan)**:** The algebras for $\mathsf{SU}(N)$, $\mathsf{SO}(N)$, $\mathsf{Sp}(N)$ and exceptional groups G_2, F_4, E_6, E_7 and E_8 are the only finite dimensional simple Lie algebras.

Theorem: All finite-dimensional representations of semisimple algebras are Hermitian.

We will shortly define what a representation is. The second theorem tells us that we can construct unitary theories based on semisimple algebras. This is why these particular algebras play a fundamental role in physics.

2 Representations

So far we have treated the group generators T^a as abstract objects. It is much more convenient if we can express them in terms of objects that we are familiar with, for example matrices. We can achieve this with representations.

Definition: Let $\mathfrak{g}$ be a Lie algebra, V be a vector space V, and $\mathfrak{gl}(V)$ denotes the set of all linear maps to itself. A representation R is a map $R : \mathfrak{g} \to \mathfrak{gl}(V)$ that satisfies

$$R([A,B]) = R(A)R(B) - R(B)R(A), \quad \forall X,Y \in \mathfrak{g}. \tag{5}$$

In simple terms, a representation maps the elements of a Lie algebra to matrices in such a way the Lie bracket operation becomes matrix commutation.

For a given Lie algebra they can be many representations. The simplest non-trivial algebra is called the fundamental (defining) representation. For $\mathsf{SU}(N)$, the fundamental representation is a map to the set of $N \times N$ Hermitian matrices. Thus this is a N-dimensional representation. A set of N fields that transform in the fundamental representation transform under an infinitesimal transformation as

$$\phi_i \mapsto \phi_i + i\alpha^a (T^a_{\text{fund}})_{ij}\phi_j, \tag{6}$$

where α^a are real numbers. The complex conjugate fields transform under the anti-fundamental representation:

$$\phi^*_i \mapsto \phi^*_i - i\alpha^a \left(T^a_{\overline{\text{fund}}}\right)_{ij}\phi^*_j. \tag{7}$$

By comparing Eqs. (10) and (11) we can see that the fundamental and the anti-fundamental representations are related to each other by

$$\left(T^a_{\overline{\text{fund}}}\right) = -(T^a_{\text{fund}}). \tag{8}$$

The default representation is the fundamental one, so from now on we drop the "fund"-subscript.

Since we will need them in the next lecture where we will discuss the spontaneous breaking of QCD, we will explicitly write the generators T^a of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ in the fundamental representation. For $\mathsf{SU}(2)$ these are proportional to the Pauli matrices σ^a:

$$T^a = \frac{\sigma^2}{2}. \tag{9}$$

In explicit form

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

For $\mathsf{SU}(3)$, the generators are given by $T^a = \lambda^a/2$ where λ^a are Gell-

Mann matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The normalization of the $\mathsf{SU}(N)$ generators in the fundamental representation is chosen such that

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}. \tag{12}$$

It is also possible to express the product of the $\mathsf{SU}(N)$ generators in the fundamental representation by

$$T^a T^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2}d^{abc}T^c + \frac{i}{2}f^{abc}T^c, \tag{13}$$

where

$$d^{abc} = 2\,\mathrm{Tr}\left(T^a\left\{T^b, T^c\right\}\right) \tag{14}$$

is a totally symmetric group invariant.

Another important representation is the adjoint representation. In this case, the vector space V on which the generators are mapped is the vector space spanned by the generators themselves. Since the number of generators is equal to the dimension of the Lie group, the dimension of the representation is the same as the dimension of the Lie group. For $\mathsf{SU}(N)$ this is $N^2 - 1$. The generators in the adjoint representation take the form

$$\left(T^a_{\text{adj}}\right)^{bc} = -if^{abc}. \tag{15}$$

In $\mathsf{SU}(2)$ these are 3×3 matrices, while in $\mathsf{SU}(3)$ these are 8×8 matrices.

The following statements are crucial to understand the Standard Model:

- Matter fields transform under the fundamental representation of a gauge group, while anti-matter fields transform under the anti-fundamental representation. Usually these are denoted by $\square$ and $\overline{\square}$ respectively.
- Gauge fields transform under the adjoint representation of a gauge group. This is denoted by adj.

3 Characterizing representations

It would be nice to have a basis-independent way of characterizing representations. The following statement provides a good starting point:

Theorem (Schur's Lemma)**:** A group element that commutes with all other group elements in any irreducible representation must be proportional to the identity matrix $\mathbb{1}$.

By using the defintion of the Lie bracket (6) we can show that

$$\left[T_R^a T_R^a, T_R^b\right] = i f^{abc} \{T_R^a, T_R^c\} = 0, \tag{16}$$

where R denotes any representation. By deriving this result we have used the facts that the structure constants f^{abc} are totally anti-symmetric and the Lie bracket definition (6) is valid for any representation with the *same* structure constants, i.e.

$$\left[T_R^a, T_R^b\right] = i f^{abc} T_R^c. \tag{17}$$

Then the Schur's Lemma implies that

$$T_R^a T_R^a = C_2(R) \cdot \mathbb{1}, \tag{18}$$

where $C_2(R)$ is called the quadratic casimir. In any representation, the generators can be chosen such that

$$\left\langle T_R^a \middle| T_R^b \right\rangle = \mathrm{Tr}\left(T_R^a T_R^b\right) = T(R)\delta^{ab}, \tag{19}$$

where $T(R)$ is called the index of the representation, and we have defined the Cartan inner product as

$$\langle A|B\rangle := \mathrm{Tr}(AB). \tag{20}$$

For $\mathsf{SU}(N)$ we get

$$T_F = \frac{1}{2}, \quad T_A = N, \tag{21}$$

where F and A stands for fundamental and adjoint representation respectively. By taking Eq. (23), setting $a = b$ and summing over a yields the relation

$$C_2(R)\dim(R) = T(R)\dim(G), \tag{22}$$

where $\dim(R)$ and $\dim(G)$ are the dimensions of the Lie group and the representation respectively. We then find

$$C_F := C_2(F) = \frac{N^2-1}{2N}, \quad C_A := C_2(A) = N. \tag{23}$$

In particular, we have $C_F = 3/4$ and $C_F = 4/3$ for $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ respectively. For any representation we can write

$$\mathrm{Tr}\left(\left[T_R^a, T_R^b\right] T_R^c\right) = i f^{abc} T(R). \tag{24}$$

Since the structure constants are the same in any representation, we can pick the fundamental and write

$$f^{abc} = -\frac{i}{T_F}\,\mathrm{Tr}\Big(\Big[T_R^a, T_R^b\Big]\,T_R^c\Big). \tag{25}$$

This means that we can always replace structure constants with the group generators. Finally, another invariant is the anomaly coefficient $A(R)$ defined by

$$\mathrm{Tr}\Big(T_R^a\Big\{T_R^b, T_R^c\Big\}\Big) = \frac{1}{2}A(R)d^{abc} = A(R)\,\mathrm{Tr}\Big(T^a\Big\{T^b, T^c\Big\}\Big). \tag{26}$$