Lecture 1b: Spontaneous Symmetry Breaking

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In this lecture we will provide an introduction to the concept of *spontaneous symmetry breaking*. This will give us the necessary toolbox to study *Chiral Lagrangians* which is an effective theory of QCD at low energies, also known as *Chiral Perturbation Theory* (χPT).

The reason why we need an effective theory to study QCD at low energies lies in the peculiar feature of the strong interactions known as the *asymptotic freedom*. This feature implies that the strength of the strong interactions becomes weaker as the energy scale increases and the corresponding length scale decreases. On the other hand, as the energy scale increases, QCD becomes strongly coupled and becomes non-perturbative.

At high energies, the degrees of freedom of QCD are the quarks and the gluons. The interactions between them can be treated via the techniques of perturbation theory. This the quark-gluon phase of the QCD. At low energies, quarks cannot exist in isolation. They form *bound states* that are called hadrons. These are further categorized as baryons that have three quarks, and mesons that have one quark and an anti-quark[1] This is the hadronic phase of the QCD, and in this phase choosing baryons and mesons as the degrees of freedom is much more convenient.

[1] Exotic quarks that are bound states of more than three quarks also exists and have been observed at the LHCb experiment.

The transition from the quark-gluon phase to the hadronic phase is a phase transition. We know roughly that this transition has happened when the temperature of the universe was $T \sim \Lambda_{\text{QCD}} \sim 300\,\text{MeV}$. But we don't know *how* this transition has happened. Despite of this difficulty, we can still obtain an *effective* description of the physics of the low-energy QCD. This is because, as we will see in the next lecture, the phase transition to the hadronic phase can be described approximately as a *spontaneous symmetry breaking*. Even though we don't know how the breaking happens, we know the *pattern* of symmetry breaking. In this case, the *Goldstone theorem* will tell us how many degrees of freedom we expect in the low energy effective theory, and how these degrees of freedom should behave.

In this lecture, we will study spontaneous symmetry breaking and the Goldstone theorem with a couple of toy models. In the next lecture, we will apply these to QCD.

1 Definition

They are two common types of symmetry breaking in particle physics:

- Spontaneous symmetry breaking: The Lagrangian is invariant under a symmetry transformation while the ground state is not.
- Explicit symmetry breaking: There was no symmetry to begin with. This term is used mainly in situations where a subset of the Lagrangian is invariant under some symmetry while additional terms are not. In this case, one says that the additional terms break the symmetry explicitly.

As we will see both kinds of symmetry breaking play a role in the study of Chiral Lagrangians.

2 Spontaneous Breaking of Discrete Symmetries

Let us consider a real scalar field ϕ with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad \text{with} \quad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4, \tag{1}$$

where $\lambda > 0$ and[2]

$$(\partial\phi)^2 := \left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) = \eta^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) \tag{2}$$

[2] Unless mentioned otherwise out metric convention is $(+,-,-,-)$.

The Lagrangian is invariant under a *discrete* $\mathbb{Z}_2$ transformation under which

$$\mathbb{Z}_2 : \phi \mapsto -\phi. \tag{3}$$

To find whether this symmetry is spontaneously broken, we need find the ground state. It corresponds to the point(s) in the field space where the potential in Eq. (1) is minimized. If $m^2 > 0$, then there is a single minimum given by

$$\phi = v_0 = 0, \quad \text{if} \quad m^2 > 0. \tag{4}$$

The corresponding vacuum state $|\Omega_0\rangle$ defined via[3]

$$v_0 = \langle\Omega_0|\Phi|\Omega_0\rangle \tag{5}$$

[3] Precisely speaking, the classical expectation v_0 is equal to the vacuum expectation value (VEV) $\langle\Omega_0|\Phi|\Omega_0\rangle$ in the classical limit $\hbar \to 0$ and when the loop contributions to VEV are ignored.

is called the vacuum expectation value (VEV). It is also invariant under the $\mathbb{Z}_2$ transformation so the symmetry is not broken. On the other hand if $m^2 < 0$, $\phi = 0$ becomes a maximum, and they are two separate minima given by

$$\phi = v_\pm = \pm\sqrt{\frac{-m^2}{\lambda}}. \tag{6}$$

These correspond to two *distinct* ground states $|\Omega_+\rangle$ and $|\Omega_-\rangle$. These have the same energy so they are degenerate. A $\mathbb{Z}_2$ transformation turns $|\Omega_+\rangle$ to $|\Omega_-\rangle$ and vice versa. So the ground state is not invariant under the $\mathbb{Z}_2$ transformation anymore, and one says that the $\mathbb{Z}_2$ symmetry is spontaneously broken.

3 Spontaneous Breaking of Continuous Symmetries

Now we consider a slightly complicated but much more interesting example. We consider a similar Lagrangian as in Eq. (1), but take a complex field Φ instead. We take[4]

[4] The superscript $*$ denotes complex conjugation.

$$\mathcal{L} = (\partial^\mu \Phi)^* (\partial_\mu \Phi) - V(\Phi) \quad \text{with} \quad V(\Phi) = m^2|\Phi|^2 + \frac{\lambda}{4}|\Phi|^4. \tag{7}$$

The Lagrangian is invariant under the $\mathsf{U}(1)$ transformation

$$\mathsf{U}(1) : \Phi \mapsto \Phi' = e^{i\alpha}\Phi, \quad \alpha \in \mathbb{R}. \tag{8}$$

This is an example of a continuous transformation since by varying α any transformation can be *continuously connected to the identity*. Since α is a real number, this is an example of a global symmetry. It would be a local (gauge) symmetry if α were a function $\alpha(x)$ instead of a number.

It is easy to check that the potential in Eq. (7) gets minimized at

$$|\Phi|^2 = \begin{cases} 0, & \text{if} \quad m^2 > 0, \\ \dfrac{-2m^2}{\lambda}, & \text{if} \quad m^2 < 0 \end{cases}. \tag{9}$$

In the first case, there is a single ground state which is invariant under a $\mathsf{U}(1)$ transformation so the symmetry is unbroken. In the second case, they are infinitely many vacua $\{\Omega_\theta\}$ defined by

$$\langle \Omega_\theta | \Phi | \Omega_\theta \rangle = \sqrt{\frac{-2m^2}{\lambda}} e^{i\theta}, \quad \theta \in [0, 2\pi). \tag{10}$$

In other words, the vacuum is infinitely degenerate. We also observe that the vacuum states do not have the symmetry of the Lagrangian anymore because under a $\mathsf{U}(1)$ transformation

$$\mathsf{U}(1) : |\Omega_\theta\rangle \mapsto |\Omega_{\theta'}\rangle \quad \text{with} \quad \theta \neq \theta'. \tag{11}$$

So the global $\mathsf{U}(1)$ symmetry is spontaneously broken.

Now we want to find the particle spectrum in the broken phase. For this we need to expand the fields around one of the vacua. Since all vacua are equivalent by the symmetry, we can choose one such that the VEV $\langle \Omega | \Phi | \Omega \rangle$ is real:

$$\langle \Omega | \Phi | \Omega \rangle = \sqrt{\frac{-2m^2}{\lambda}} =: v. \tag{12}$$

We can expand Φ around v by $\Phi = v + \widetilde{\Phi}$ where $\widetilde{\Phi}$ is a new complex field, but a more convenient way is to introduce two *real* scalar fields σ and π and write

$$\Phi(x) = \left(v + \frac{1}{\sqrt{2}}\sigma(x) \right) \exp\left\{ \frac{i\pi(x)}{F} \right\}, \tag{13}$$

where F is a constant with mass dimension one that we shall fix shortly. Substituting this expansion into the Lagrangian (7) yields

$$\mathcal{L} = \frac{1}{2}(\partial\sigma)^2 + \frac{v^2}{F^2}(\partial\pi)^2 + \left(\frac{\sqrt{2}v}{F^2}\sigma + \frac{1}{2F^2}\sigma^2\right)(\partial\pi)^2 + \mathcal{L}_\sigma, \tag{14}$$

where

$$\mathcal{L}_\sigma = \frac{\lambda v^4}{4} - \left(\frac{1}{2}\lambda v^2\sigma^2 + \frac{1}{2\sqrt{2}}\lambda v\sigma^3 + \frac{1}{16}\lambda\sigma^4\right). \tag{15}$$

We can choose F such that the kinetic term for the π field is canonically normalized which gives $F = \sqrt{2}v$. Our final result is

$$\mathcal{L} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\pi)^2 + \left(\frac{\sigma}{F} + \frac{\sigma^2}{2F^2}\right)(\partial\pi)^2 + \mathcal{L}_\sigma. \tag{16}$$

Let us see what we can learn from this Lagrangian. Apart from the constant term $\lambda v^4/4$ which is irrelevant we have the following:

- A real field σ with the mass $m_\sigma^2 = \lambda v^2 = \lambda F^2/2$ that has cubic and quartic self-interactions.
- A real field π which is *massless* that interacts *derivatively* with the σ field.

Easiest way to understand the physical meanings of the fields σ and π is to visually inspect the plot of the potential $V(\Phi)$ with $m^2 < 0$ in Figure 1. Both σ and π are excitations of the complex field Φ around the minimum $v = \sqrt{-2m^2/\lambda}$. The σ field corresponds to the excitations around the radial direction. It has non-zero mass, becuse energy is needed to get displaced from the minimum in the radial direction. On the other hand, the π field corresponds to excitations in the angular direction. Since it doesn't cost energy to move in the angular direction, the π field is massless. Due to this picture, the σ and π fields are called radial and angular field respectively. The π is also called a Nambu-Goldstone boson (NGB).

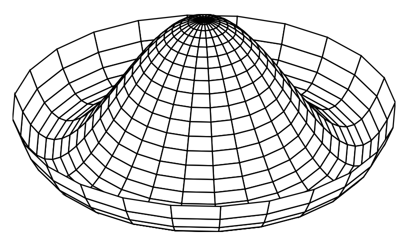

Figure 1: A plot of the potential $V(\Phi)$ with $m^2 < 0$.

Although the $\mathsf{U}(1)$ symmetry appears to be broken when $m^2 < 0$, it has not disappeared from the Lagrangian. It is still realized, albeit in a different way. To see this, note that under a $\mathsf{U}(1)$ transformation $\Phi(x) \mapsto \Phi'(x) = e^{i\alpha}\Phi(x)$, the expansion (13) becomes

$$\Phi'(x) = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)\exp\left\{\frac{i(\pi(x) + \alpha F)}{F}\right\}, \tag{17}$$

which means that in the broken phase, the original $\mathsf{U}(1)$ symmetry is realized as a shift symmetry:

$$\pi(x) \mapsto \pi(x) + \alpha F. \tag{18}$$

This shift symmetry has two important implications:

- It forbids a mass term π^2 for the Goldstone bosons.
- It implies that the Goldstone bosons couple to other fields only derivatively.

We can see that both of these properties are realized in the broken phase Lagrangian that we have derived in Eq. (16). These properties will help us tremendously in the study of Chiral Lagrangians.

4 Multiple scalar fields

Finally, we consider a Lagrangian with N real scalar fields $\{\phi_i\}_{i=1}^N$:

$$\mathcal{L} = \frac{1}{2}\sum_n (\partial\phi_n)^2 - V(\{\phi_n\}), \quad V(\{\phi_n\}) = \frac{1}{2}\sum_n \phi_n\phi_n + \frac{\lambda}{4}\sum_n (\phi_n\phi_n)^2. \tag{19}$$

We can write this Lagrangian in a more compact form by introducing a vector $\boldsymbol{\phi}$ whose components are N scalar fields ϕ_n. Then

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\boldsymbol{\phi})\cdot(\partial_\mu\boldsymbol{\phi}) - \frac{1}{2}m^2|\boldsymbol{\phi}|^2 - \frac{\lambda}{4}|\boldsymbol{\phi}|^4, \tag{20}$$

where $|\boldsymbol{\phi}|^2 = \boldsymbol{\phi}\cdot\boldsymbol{\phi}$. This Lagrangian is invariant under N-dimensional rotations of the vector $\boldsymbol{\phi}$; namely the symmetry group is $\mathsf{O}(N)$. Again, the minimum, which we denote with $\mathbf{v}$, depends on the sign of m^2 and given by

$$v^2 := |\mathbf{v}|^2 = \begin{cases} 0, & \text{if} \quad m^2 > 0, \\ -m^2/\lambda, & \text{if} \quad m^2 < 0 \end{cases} . \tag{21}$$

For $m^2 < 0$, the vacuum is again infinitely degenerate. Any vector $\mathbf{v}$ whose length is $v = \sqrt{-m^2/\lambda}$ is a vacuum state. Under an $\mathsf{O}(N)$ rotation $\mathbf{v} \mapsto \mathbf{v}'$ with $\mathbf{v}' \neq \mathbf{v}$ so the $\mathsf{O}(N)$ symmetry is spontaneously broken. To find the mass spectrum in the broken phase, we expand the potential $V(\{\phi_n\})$ around *any* vacua $\mathbf{v}$ satisfying $|\mathbf{v}|^2 = -m^2/\lambda$. This yields

$$V(\boldsymbol{\phi}) \simeq V(\mathbf{v}) + \sum_n \left.\frac{\partial V}{\partial\phi_n}\right|_{v_n} (\phi_n - v_n) + \sum_{n,m} M_{mn}(\phi_n - v_n)(\phi_m - v_m) \tag{22}$$

up to second order in $(\phi_n - v_n)$. The first term is a constant and hence irrelevant. The second term vanishes since $\mathbf{v}$ minimizes the potential. The second term contains the mass matrix M_{mn} given by

$$M_{mn} := \left.\frac{\partial^2 V}{\partial\phi_n\partial\phi_m}\right|_{\mathbf{v}}. \tag{23}$$

A simple calculation yields

$$M_{mn} = 2\lambda v_n v_m. \tag{24}$$

The mass spectrum is given by the eigenvalues of this matrix. The easiest way to calculate them is to invoke the fact that we can switch from one vacuum state to the other by a $\mathsf{O}(N)$ transformation. So we can choose $\mathbf{v}$ to be

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix}. \tag{25}$$

For this choice, the mass matrix becomes diagonal with a single non-zero eigenvalue

$$\mu^2 := 2\lambda v^2 = -\frac{m^2}{2} > 0, \tag{26}$$

and $N-1$ zero eigenvalues. This implies the existence of $N-1$ massless states, i.e. Goldstone bosons.

Note that the vacuum state is not invariant under $(O)(N)$, but it is invariant under a *smaller* symmetry group $(O)(N-1)$[5].One says that the symmetry is spontaneously broken to the subgroup $\mathsf{O}(N-1) \subset \mathsf{O}(N)$.

[5] This fact can most easily be seen from the Eq. (25). An $(O)(N-1)$ transformation acts on the first zero $N-1$ elements, and leaves $\mathbf{v}$ invariant.

5 Goldstone theorem

We now state the celebrated Goldstone theorem.

Theorem (Goldstone)**:** In the case of spontaneously broken continuous global symmetry, the spectrum of physical particles must contain one particle of zero mass and spin for each broken symmetry.

Multiple proofs can be found in the Volume 2 of the Weinberg's classic QFT book.[6] This theorem allows us to determine the number of massless states in the broken phase of the previous example without making any calculation. To see this we need to remember that the dimension of the $\mathsf{O}(N)$ group is

[6] Steven Weinberg (Aug. 2013). *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press.

$$\dim \mathsf{O}(N) = \frac{1}{2}N(N-1). \tag{27}$$

Since the symmetry is broken to the subgroup $\mathsf{O}(N) \subset \mathsf{O}(N-1)$, the number of broken generators is

$$\begin{aligned} \#\text{ broken generators} &= \dim \mathsf{O}(N) - \dim \mathsf{O}(N-1) \\ &= \frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) \\ &= N-1, \end{aligned} \tag{28}$$

which agrees with what we have found in the previous section. We will see the real power of the Goldstone theorem when we discuss the symmetry breaking in QCD, since there we won't have any microscopic description of the symmetry breaking process.

References

Weinberg, Steven (Aug. 2013). *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press. ISBN: 978-1-139-63247-8, 978-0-521-67054-8, 978-0-521-55002-4.